Cramé r's estimate for a reflected Lé vy process*

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Abstract

The natural analogue for a Lévy process of Cramér's estimate for a reflected random walk is a statement about the exponential rate of decay of the tail of the characteristic measure of the height of an excursion above the minimum. We establish this estimate for any Lévy process with finite negative mean which satisfies Cramér's condition, and give an explicit formula for the limiting constant. Just as in the random walk case, this leads to a Poisson limit theorem for the number of "high excursions".

Key words and phrases. Maximum of reflected process, maximal segmental score, Poisson limit theorem, high excursions.

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1. Introduction

The reflected process $R = (R_n, n \geq 0)$ formed from a random walk $S = (S_n, n \geq 0)$ by setting

$$R_n = S_n - I_n \text{ where } I_n = \min_{i \leq n} S_i, \ n \geq 0,$$

arises in many areas of applied probability, including queuing theory, risk theory, and mathematical genetics; see e.g. [1], [2], [6], and [8]. In these applications, the current maximum of R,

$$M_n^{(R)} = \max_{j \leq n} R_j = \max_{0 \leq i \leq j \leq n} \{S_j - S_i\},$$

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is of particular interest; in the genetics context this is called the maximal segmental score. The segments here correspond to excursions of the random walk above its minimum, and more generally the behaviour of

$$N(y,n) = \#\{\text{excursions completed by time } n \text{ whose heights exceed } y\}$$

is important. A classical result in [7] asserts that if the random walk has a finite negative drift and Cramér's condition is satisfied then $N(y,n)$ has a limiting Poisson distribution when $n, y \to \infty$ in such a way that ne^{-y} converges to a positive constant. In [8] extensions of this result to processes other than random walks were given, including compound Poisson processes and Brownian motion with negative drift. We show here that these extensions actually hold when the underlying process is any Lévy process with finite negative drift which satisfies Cramér's condition.

2. The Random Walk case

Let S be any random walk with finite negative mean which satisfies Cramer's condition, viz $E(e^{\gamma S_1}) = 1$ for some $\gamma \in (0,\infty)$. Let $h_i, i = 1, ..$ denote the height of the ith excursion above the minimum, i.e.

$$h_i = \max_{0 \le n \le T_i - T_{i-1}} \{S_{T_{i-1}+n} - S_{T_{i-1}}\},$$

where T_i is the ith strict descending ladder time, with $T_0 = 0$, and let M_∞ denote the all-time maximum of the walk. Also write T_i^+, $H_i^+ = S(T_i^+)$ for the weak increasing ladder times and heights and $H_i = |S(T_i)|$ for the decreasing ladder heights. Then Cramér's famous estimate states that

$$\lim_{x\to\infty} e^{\gamma x} P(M_\infty > x) = C = \frac{P(H_1^+ < \infty)}{\gamma E(H_1^+ e^{\gamma H_1^+}; H_1^+ < \infty)}. \tag{1}$$

(See e.g. p 413 of [6].) Obviously the h_i are independent, identically distributed and it is easy to deduce from the above that, in the non-lattice case,

$$e^{\gamma x} P(h_1 > x) \to K := C\{1 - E(e^{-\gamma H_1})\}. \tag{2}$$

To see this, observe the identity

$$P(M_\infty > x) = P(h_1 > x) + \int_0^\infty P(h_1 \le x, H_1 \in dy) P(\tilde{M}_\infty > x + y), \tag{3}$$

where $\tilde{M}_\infty$ is an independent copy of M_∞, multiply by $e^{\gamma x}$, and let $x \to \infty$; this argument is due to Iglehart [7].

Now introduce the finite constant $\alpha = ET_1$, so that the strong law implies that

$$\frac{\sup\{i : T_i \leq n\}}{n} \stackrel{a.s.}{\to} \frac{1}{\alpha} \text{ as } n \to \infty.$$

Using (2) it is then easy to see that if n and and $y \to \infty$ in such a way that $ne^{-\gamma y} \to \alpha\lambda/K$ then the number of the h_i which exceed y and occur by time n has a limiting Poisson(λ) distribution.

3. The Lévy process case

Now let X be any Lévy process with $EX_1 \in (-\infty, 0)$ which satisfies Cramér's condition, viz

$$E(e^{\gamma X_1}) = 1 \text{ for some } \gamma \in (0, \infty),$$

and define Y, the process reflected in its infimum, by

$$Y_s = X_s - I_s, \text{ where } I_s = \inf_{0 \leq u \leq s} X_u, \; s \geq 0.$$

Let $L = (L_t, t \geq 0)$ and $L^{-1} = (L_t^{-1}, t \geq 0)$ denote the local time process of Y at 0 and its right continuous inverse, and write

$$\varepsilon_t(u) = Y(L_t^{-1} + u) - Y(L_t^{-1}), \; u \geq 0, \text{ and } \xi_t = \inf\{u : \varepsilon_t(u) \leq 0\}.$$

Then $\varepsilon_t = (\varepsilon_t(u), 0 \leq u < \xi_t)$ is the excursion above the minimum at local time t, and

$$h_t = \sup(\varepsilon_t(u), 0 \leq u < \xi_t)$$

is the height of this excursion. Thus the Poisson point process $(h_t, t \geq 0)$ is the continuous time version of $(h_i, i \geq 1)$ for a random walk . If η denotes the characteristic measure of $(h_t, t \geq 0)$ then the statement which corresponds to (2) in this context is that

$$e^{\gamma x}\eta((x, \infty)) \to K^* \text{ as } x \to \infty. \tag{4}$$

Our aim is to establish (4), determine the constant K^*, and deduce that the number of excursions of Y away from 0 with heights exceeding y which take place by time t satisfies a Poisson limit theorem. Of course the reason that this is potentially more difficult than the random walk case is that there may be an infinite number of excursions in any finite time interval.

The starting point, naturally, is the following known Lévy process version of (1); (see [4]): with $S_\infty = \sup_{0 \le t < \infty} X_t$ denoting the all-time supremum and $H^+ = (H_t^+, t \ge 0)$ the increasing ladder-height process,

$$e^{\gamma x} P(S_\infty > x) \to C^* = \frac{\beta}{\gamma m} \text{ as } x \to \infty, \tag{5}$$

$$\text{where } \beta = -\log P(H_1^+ < \infty) \text{ and } m = E(H_1^+ e^{\gamma H_1^+}; H_1^+ < \infty). \tag{6}$$

Actually it is easy, by applying the random walk argument to the independent, identically distributed sequence of excursions heights which exceed some fixed $\delta > 0$, to deduce from (5) that (4) holds, but with a value of K^* which apparently depends on δ. Thus the proof of the following theorem, which is quite delicate, is essentially a matter of justifying an interchange of limits. The inverse local time process $L^{-1} = (L_t^{-1}, t \ge 0)$ is the Lévy version of the descending ladder time process, and the corresponding ladder height process is defined by

$$H_t = |X(L_t^{-1})| = |I(L_t^{-1})|, \text{ where } I_t = \inf_{0 \le s \le t} X_s.$$

Theorem 1. *(i) Let $\phi(\theta) = -\log E(e^{-\theta H_1})$ denote the exponent of the subordinator H. Then, as $x \to \infty$,*

$$e^{\gamma x} \eta((x, \infty)) \to K^* := \phi(\gamma) C^* = \frac{\phi(\gamma)\beta}{\gamma m}.$$

(ii) Introduce the finite constant $\alpha^* = EL_1^{-1}$, and let $N(y,t)$ denote the number of excursions of Y with heights exceeding y which occur by time t. Let $t, y \to \infty$ in such a way that $te^{-\gamma y} \to \alpha^* \lambda / K^*$. Then $N(y,t)$ has a limiting Poisson(λ) distribution.

Proof. Write $S_t = \sup_{0 \le u \le t} X_u$ and $\hat{S}_t = S(L_t^{-1})$. Then applying the Markov property at the stopping time L_t^{-1} we see that

$$P(S_\infty \le x) = P\{(\hat{S}_t \le x) \cap (\tilde{S}_\infty \le x + H_t)\},$$

where $\tilde{S}_\infty$ is independent of $\hat{S}_t$ and H_t and has the distribution of S_∞. Note however that H_t and $\hat{S}_t$ are dependent. Subtracting the term $P(\hat{S}_t \le x)P(S_\infty \le x)$ gives

$$P(\hat{S}_t > x) P(S_\infty \le x) = P\{(\hat{S}_t \le x) \cap (x < \tilde{S}_\infty \le x + H_t)\}. \tag{7}$$

Our first step is to examine the behaviour of $e^{\gamma x}\ P(\hat{S}_t > x)$ as $x \to \infty$ for fixed t. Note first that in view of (5) and the fact that $P(\hat{S}_t > x) \to 0$ as $x \to \infty$ for fixed t,

$$e^{\gamma x} P\{(\hat{S}_t > x) \cap (x < \tilde{S}_\infty \le x + H_t)\} \le e^{\gamma x} P(\hat{S}_t > x) P(\tilde{S}_\infty > x) \to 0.$$

Thus as $x \to \infty$, for each fixed t,

$$\begin{aligned} P(x < \tilde{S}_\infty \le x + H_t) &= e^{\gamma x} P\{(\hat{S}_t > x) \cap (x < \tilde{S}_\infty \le x + H_t)\} + o(1) & (8) \\ &= e^{\gamma x} P(\hat{S}_t > x) P(S_\infty \le x) + o(1) & (9) \\ &= e^{\gamma x} P(\hat{S}_t > x) + e^{\gamma x} P(\hat{S}_t > x) P(S_\infty > x) + o(1) & (10) \\ &= e^{\gamma x} P(\hat{S}_t > x) + o(1). & (11) \end{aligned}$$

Next recall from [4] that we can write $e^{\gamma x} P(S_\infty \in dx) = \beta U(dx)$, where U is the renewal measure corresponding to a distribution on $[0, \infty)$ with the finite mean m given in (6). So we can write

$$e^{\gamma x} P(x < \tilde{S}_\infty \le x + H_t) = \beta E \left(\int_0^{H_t} e^{-\gamma y} U(x + dy) \right) := \beta E Z_t(x). \qquad (12)$$

Since $mU(x + dy)$ converges weakly to Lebesgue measure, we see that, for each fixed ω and t,

$$Z_t(x, \omega) \to m^{-1} \int_0^{H_t(\omega)} e^{-\gamma y} dy = \frac{1 - e^{-\gamma H_t(\omega)}}{m\gamma} \text{ as } x \to \infty. \qquad (13)$$

Also, using the subadditivity of U and Erickson's bounds (see [5]), we get

$$Z_t(x, \omega) \le U(x + H_t(\omega)) - U(x) \le U(H_t(\omega)) \le \frac{2H_t(\omega)}{m}.$$

Since $EX_1 \in (-\infty, 0)$ we have $EH_t < \infty$ for any fixed t, so by dominated convergence we conclude from (8), (12), and (13) that

$$\lim_{x \to \infty} e^{\gamma x} P(\hat{S}_t > x) = \beta E \lim_{x \to \infty} Z_t(x) = \frac{\beta E(1 - e^{-\gamma H_t})}{m\gamma} = \frac{\beta(1 - e^{-t\phi(\gamma)})}{m\gamma}. \qquad (14)$$

To connect this to the asymptotic behaviour of $e^{\gamma x}\overline{\eta}(x)$, where $\overline{\eta}(x) = \eta((x, \infty))$, we need the following observation; the event $\hat{S}_t \le x$ occurs if and only if each excursion at local time s, for all $s \le t$, has height $\le H_s + x$. Hence, by a standard application of the compensation formula, (see [3], p7), we have

$$t^{-1} P(\hat{S}_t > x) = t^{-1} E(1 - \exp\{-\int_0^t \overline{\eta}(x + H_s) ds\}). \tag{15}$$

Using the bound $1 - e^{-x} \le x$ in this shows that for any $t > 0$

$$t^{-1} e^{\gamma x} P(\hat{S}_t > x) \le t^{-1} e^{\gamma x} E \int_0^t \overline{\eta}(x + H_s) ds \le e^{\gamma x} \overline{\eta}(x).$$

From (14) we see that

$$\liminf_{x\to\infty} e^{\gamma x} \overline{\eta}(x) \ge \lim_{t\downarrow 0} \lim_{x\to\infty} t^{-1} e^{\gamma x} P(\hat{S}_t > x) = \lim_{t\downarrow 0} \frac{\beta(1 - e^{-t\phi(\gamma)})}{m\gamma t} = K^*.$$

Also, using the bound $1 - e^{-x} \ge x - x^2/2$ in (15) gives, for any fixed $\varepsilon > 0$ and all $t > 0$,

$$\begin{aligned} t^{-1} e^{\gamma x} P(\hat{S}_t > x) &\ge t^{-1} e^{\gamma x} \left(E \int_0^t \overline{\eta}(x + H_s) ds - \frac{1}{2} E \left\{ \int_0^t \overline{\eta}(x + H_s) ds \right\}^2 \right) \\ &\ge e^{\gamma x} \left(E \overline{\eta}(x + H_t) - \frac{t}{2} \overline{\eta}(x)^2 \right) \\ &\ge e^{\gamma x} \overline{\eta}(x + \varepsilon) P(H_t \le \varepsilon) - \frac{t}{2} e^{\gamma x} \overline{\eta}(x)^2. \end{aligned}$$

Rearranging, using (14), the fact that $P(H_t \le \varepsilon) \to 1$ as $t \downarrow 0$, and noting that the final term above is $o(\limsup_{x\to\infty} e^{\gamma x} \overline{\eta}(x))$ we see that

$$\begin{aligned} \limsup_{x\to\infty} e^{\gamma x} \overline{\eta}(x) &= e^{\gamma\varepsilon} \limsup_{x\to\infty} e^{\gamma x} \overline{\eta}(x + \varepsilon) \\ &\le e^{\gamma\varepsilon} \lim_{t\downarrow 0} \lim_{x\to\infty} e^{\gamma x} t^{-1} P(\hat{S}_t > x) / P(H_t \le \varepsilon) \\ &= e^{\gamma\varepsilon} K^*. \end{aligned}$$

Since ε is arbitrary, the proof of (i) is complete.

For (ii), note that standard properties of Poisson point processes show that $N(y, t)$ has a Poisson distribution with parameter $(L_t \overline{\eta}(y))$. It follows from the strong law for subordinators, ([3], p92), that $L_t \sim t/\alpha^*$ as $t \to \infty$, so the conclusion follows. ■

Remark 2. *It is easy to adapt our arguments to extend the convergence in (ii) of Theorem 1 to joint convergence in law for different values of λ, and this leads to a process version of our result. (We are grateful to a referee for this observation.)*

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