Structures with lots of symmetry

(one of the things Bob likes to do when not doing semigroup theory)

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Why?

An advertisement

R. Gray, D. Macpherson, C. E. Praeger, G. F. Royle.
"Set homogeneous directed graphs" *J. Comb. Theory Ser. B* (in press)

It is an interesting area

- Provides a meeting-point of ideas from combinatorics, model theory, and permutation group theory.
- At a workshop in Leeds on this topic a few weeks ago several speakers mentioned semigroups in their talks...

Outline

Homogeneous relational structures

Definition

A relational structure M is homogeneous if every isomorphism between finite substructures of M can be extended to an automorphism of M.

Relational structures

- a relational structure consists of a set A, and some relations $R_1, \ldots, R_m$ (can be unary, binary, ternary, ...)
- an (induced) substructure is obtained by taking a subset $B \subseteq A$ and keeping only those relations where all entries in the tuple belong to B
- an isomorphism is a "structure preserving" mapping (i.e. a bijection ϕ such that ϕ and ϕ^{-1} are both homomorphisms)

Example

A graph Γ is a structure $(V\Gamma, \sim)$ where $V\Gamma$ is a set, and $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$.

Examples of homogeneous structures

X - a pure set

- automorphism group is the full symmetric group where any partial permutation can be extended to a (full) permutation

$(\mathbb{Q}, \leq)$ - the rationals with their usual ordering

- the automorphisms are the order-preserving permutations
- isomorphisms between finite substructures can be extended to automorphisms that are piecewise-linear

Rado's countable random graph R

- if we choose a countable graph at random (edges independently with probability $\frac{1}{2}$), then with probability 1 it is isomorphic to R

Some history

Origins

- The notion of homogeneous structure goes back to the fundamental work of Fraïssé (1953)
- Fraïssé proved a theorem which helps us determine if a countable structure is homogeneous, using the ideas of:
  - age - the finite substructures they embed, and
  - amalgamation property - the way that they can be glued together

Homogeneous structures are nice because they:

- have "lots of" symmetry;
- often have rich and interesting automorphism groups;
- give examples of "nice" $\aleph_0$-categorical structures (precisely those that have quantifier elimination).

(M is $\aleph_0$-categorical if all countable models of the first-order theory of M are isomorphic to M.)

Classification results

For certain families of relational structure, those members that are homogeneous have been completely determined.

Some classification results

	Finite	Countably infinite
Posets	(trivial)	Schmerl (1979)
Tournaments	Woodrow (1976)	Lachlan (1984)
Graphs	Gardiner (1976)	Lachlan & Woodrow (1980)
Digraphs	Lachlan (1982)	Cherlin (1998)

Set-homogeneity

Definition

A relational structure M is set-homogeneous if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $Ug = V$.

- It is a concept originally due to Fraïssé and Pouzet.
- The permutation group-theoretic weakening

  homogeneous $\rightsquigarrow$ set-homogeneous

  relates to the model-theoretic weakening

  elimination of quantifiers $\rightsquigarrow$ model complete.

- Droste et al. (1994) - proved a set-homogeneous analogue of Fraïssé's theorem, where the amalgamation property is replaced by something called the twisted amalgamation property.

Set-homogeneity vs homogeneity

- Clearly if M is homogeneous then M is set-homogeneous.
- What about the converse?

General question

How much stronger is homogeneity than set-homogeneity?

Set-homogeneous finite graphs

Ronse (1978)

...proved that for finite graphs homogeneity and set-homogeneity are equivalent.

- He did this by classifying the finite set-homogeneus graphs and then observing that they are all, in fact, homogeneous.
- This generalised an earlier result of Gardiner, classifying the finite homogeneous graphs.

Enomoto (1981)

...gave a direct proof of the fact that for finite graphs set-homogeneous implies homogeneous.

- this avoids the need to classify the set-homogeneous graphs
- the set-homogeneous classification can then be read off from Gardiner's result

Some graph theoretic terminology and notation

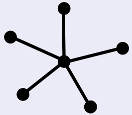
Definition

$\Gamma = (V\Gamma, \sim)$ - a graph

So $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$

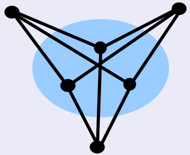
- Let v be a vertex of Γ. Then the neighbourhood $\Gamma(v)$ of v is the set of all vertices adjacent to v. So

$$\Gamma(v) = \{w \in V\Gamma : w \sim v\}$$



- For $X \subseteq V\Gamma$ we define

$$\Gamma(X) = \{w \in V\Gamma : w \sim x \, \forall x \in X\}$$



Enomoto's argument

Lemma (Enomoto's lemma)

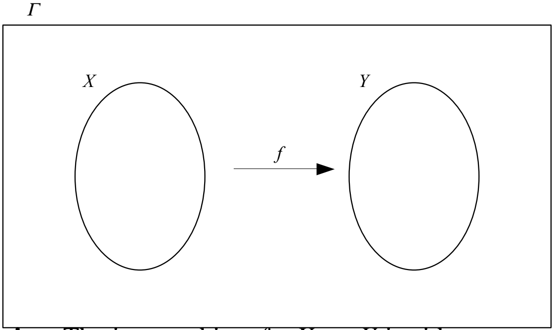
Let Γ be a finite set-homogeneous graph and let U and V be induced subgraphs of Γ. If $U \cong V$ then $|\Gamma(U)| = |\Gamma(V)|$.

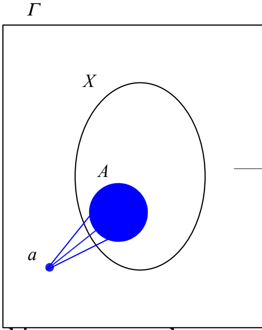
Proof.

- Let $g \in \mathrm{Aut}(\Gamma)$ such that $Ug = V$.
- Then $(\Gamma(U))g = \Gamma(V)$.
- In particular $|\Gamma(U)| = |\Gamma(V)|$.

Enomoto's argument

Γ - finite set-homogeneous graph $\quad X, Y$ - induced subgraphs
$f : X \to Y$ an isomorphism





Claim: The isomorphism $f : X \to Y$ is either an automorphism, or extends to an isomorphism $f' : X' \to Y'$ where $X' \supsetneq X$ and $Y' \supsetneq Y$.
Proof of claim.

- Choose $a \in \Gamma \setminus X$ with $|\Gamma(a) \cap X|$ as large as possible.
- Choose $d \in \Gamma \setminus Y$ with $|\Gamma(d) \cap Y|$ as large as possible.
- Suppose $|\Gamma(a) \cap X| \geq |\Gamma(d) \cap Y|$ (the other possibility is dealt with

Set-homogeneous digraphs

Question: Does Enomoto's argument apply to other kinds of structure?

Definition (Digraphs)

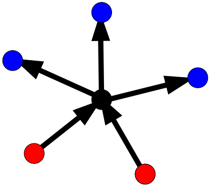
A digraph D consists of a set VD of vertices together with an irreflexive antisymmetric binary relation $\to$ on VD.

Definition (in- and out-neighbours)

A vertex $v \in VD$ has a set of in-neighbours and a set of out-neighbours

$$D^+(v) = \{w \in VD : v \to w\}, \quad D^-(v) = \{w \in VD : w \to v\}.$$

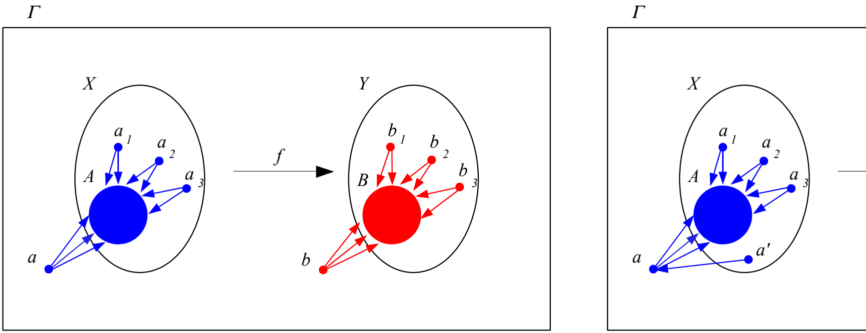
A vertex with red in-neighbours and blue out-neighbours



Enomoto's argument for digraphs

D - finite set-homogeneous digraph $\quad X, Y$ - induced subdigraphs

$f : X \to Y$ an isomorphism



- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.
- We do not know how b is related to the vertices in the set $Y \setminus B$.
  So f' may not be an isomorphism.

Enomoto's argument for digraphs

The key point:

- For graphs, given $u, v \in V\Gamma$ there are 2 possibilities

$$u \sim v \quad \text{or} \quad u \parallel v \ \text{(meaning that } u \ \& \ v \text{ are unrelated).}$$

- For digraphs, given $u, v \in VD$ there are 3 possibilities

$$u \to v \quad \text{or} \quad v \to u \quad \text{or} \quad u \parallel v.$$

However, the argument does work for tournaments:

Definition

A tournament is a digraph where for any pair of vertices u, v either $u \to v$ or $v \to u$.

Corollary

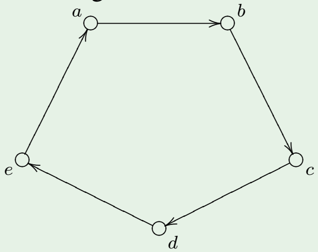
Let T be a finite tournament. Then T is homogeneous if and only if T is set-homogeneous.

A non-homogeneous example

Example

Let D_n denote the digraph with vertex set $\{0, \ldots, n-1\}$ and just with arcs $i \to i+1 \pmod{n}$.

The digraph D_5 is set-homogeneous but is not homogeneous.



- $(a, c) \mapsto (a, d)$ gives an isomorphism between induced subdigraphs that does not extend to an automorphism
- However, there is an automorphism sending $\{a, c\}$ to $\{a, d\}$.

Finite set-homogeneous digraphs

Question

How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

Theorem (RG, Macpherson, Praeger, Royle (2011))

Let D be a finite set-homogeneous digraph. Then either D is homogeneous or it is isomorphic to D_5.

Proof.

- Carry out the classification of finite set-homogeneous digraphs.
- By inspection note that D_5 is the only non-homogeneous example. $\square$

Symmetric-digraphs (s-digraphs)

A common generalisation of graphs and digraphs

Definition (s-digraph)

- An s-digraph is the same as a digraph except that we allow pairs of vertices to have arcs in both directions.
- So for any pair of vertices u, v exactly one of the following holds:

$$u \to v, \qquad v \to u, \qquad u \leftrightarrow v, \qquad u \parallel v.$$

- Formally we can think of an s-digraph as a structure M with two binary relations $\to$ and $\sim$ where
  - $\sim$ is irreflexive and symmetric (and corresponds to $\leftrightarrow$ above)
  - $\to$ is irreflexive and antisymmetric
  - $\sim$ and $\to$ are disjoint
- A graph is an s-digraph (where there are no $\to$-related vertices)
- A digraph is an s-digraph (where there are no $\sim$-related vertices)

Classifying the finite homogeneous s-digraphs

- Lachlan (1982) classified the finite homogeneous s-digraphs

To state his result we need the notions of

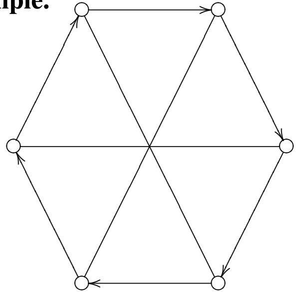
- complement
- compositional product

Finite homogeneous s-digraphs

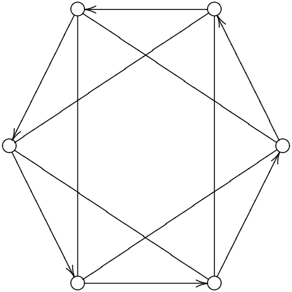
Definition (Complement)

If M is an s-digraph, then $\bar{M}$, the complement, is the s-digraph with the same vertex set, such that $u \sim v$ in $\bar{M}$ if and only if they are unrelated in M, and $u \to v$ in $\bar{M}$ if and only if $v \to u$ in M.

Example.



M



$\overline{M}$

Finite homogeneous s-digraphs

Definition (Composition)

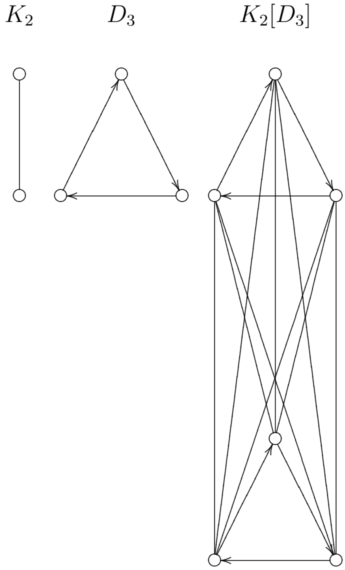
If U and V are s-digraphs, the compositional product $U[V]$ denotes the s-digraph which is

"$|U|$ copies of V"

Vertex set = $U \times V$

$\rightarrow$ relations are of form $(u, v_1) \rightarrow (u, v_2)$ where $v_1 \rightarrow v_2$ in V, or of form $(u_1, v_1) \rightarrow (u_2, v_2)$ where $u_1 \rightarrow u_2$ in U,

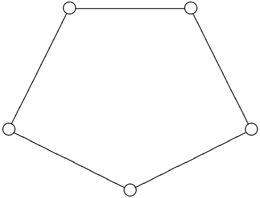
Similarly for $\sim$.



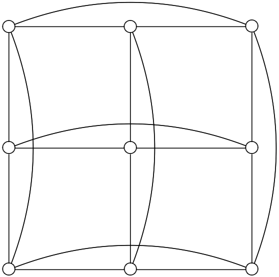
Some finite homogeneous s-digraphs

Sporadic examples

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs



$C_5 \in \mathcal{L}$



$K_3 \times K_3 \in \mathcal{L}$

Some finite homogeneous s-digraphs

Sporadic examples

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
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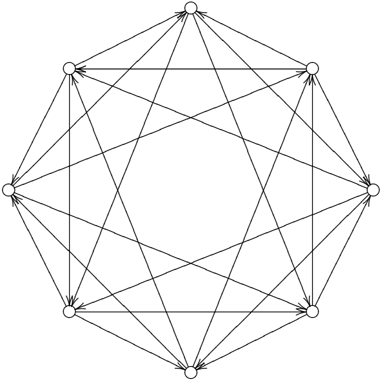
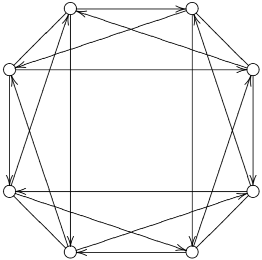


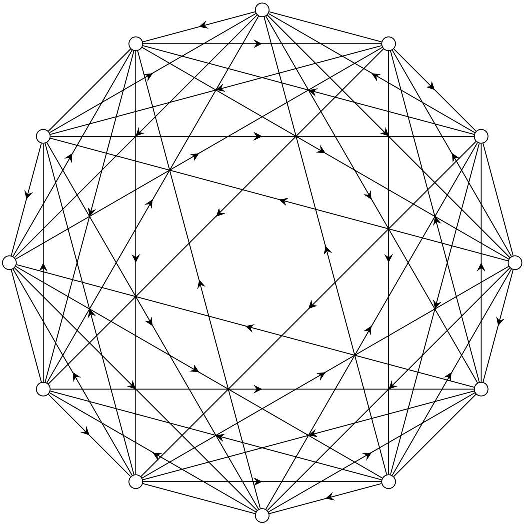
$H_0 \in \mathcal{A}$



$H_1 \in \mathcal{S}$

Some finite homogeneous s-digraphs

Sporadic examples



$H_2 \in \mathcal{S}$

Lachlan's classification

$\mathcal{L}$ - finite homogeneous graphs, $\mathcal{A}$ - finite homogeneous digraphs,
$\mathcal{S}$ - finite homogeneous s-digraphs

Theorem (Lachlan (1982))

Let M be a finite s-digraph. Then

Gardiner

(i) $M \in \mathcal{L} \Leftrightarrow M$ or $\bar{M}$ is one of: C_5, $K_3 \times K_3$, $K_m[\bar{K}_n]$ (for $1 \leq m, n \in \mathbb{N}$);

Lachlan

(ii) $M \in \mathcal{A} \Leftrightarrow M$ is one of: D_3, D_4, H_0, $\bar{K}_n$, $\bar{K}_n[D_3]$, or $D_3[\bar{K}_n]$, for some $n \in \mathbb{N}$ with $1 \leq n$;

(iii) $M \in \mathcal{S} \Leftrightarrow M$ or $\bar{M}$ is isomorphic to an s-digraph of one of the following forms. $K_n[A], A[K_n]$, L, $D_3[L]$, $L[D_3]$, H_1, H_2, where $n \in \mathbb{N}$ with $1 \leq n$, $A \in \mathcal{A}$ and $L \in \mathcal{L}$.

Set-homogeneous s-digraphs

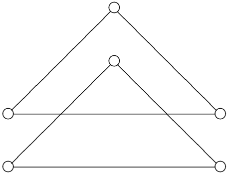
Theorem (RG, Macpherson, Praeger, Royle (2011))

The finite s-digraphs that are set-homogeneous but not homogeneous are:

Infinite families (with $n \in \mathbb{N}$)

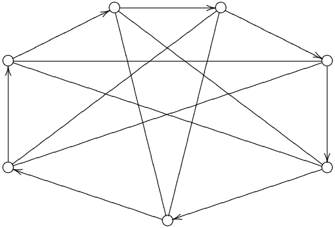
(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) J_n

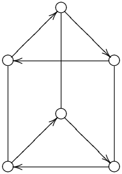


J_2

3 Sporadic examples



E_7



F_6

A monster 27-vertex sporadic example H_3

Structure of the proof

Part 1: The hunt

Build a catalogue of small examples/families of examples.

Part 2: The induction

Argue by induction on $|D|$ that every example is in our list, making use of

Lemma. *Let D be a set-homogeneous digraph and $v \in D$. Then both $D^+(v)$ and $D^-(v)$ induce (smaller) set-homogeneous digraphs.*

Case analysis: each case leading to either (a) contradiction (b) forces the structure of an example in our list.

Why might a semigroup theorist be interested?

Inverse semigroups

- Relationship between partial and global symmetries
  - Factorizable inverse monoids (nice looking survey (Fitzgerald, 2010)).
- James East says there is a variation of factorizable which gives the analogous class but for set-homogeneity.

Semigroups (full endomorphism monoids)

- Maltcev, Mitchell, Péresse, Ruškuc: Bergman property, Sierpiński rank.
- Bodirsky and Pinsker: reducts of the random graph.
- Bonato, Delić, Dolinka, Mašulović, Mudrinski: structural properties.
- Lockett, Truss: generic endomorphisms of homogeneous structures.

Homomorphism homogeneity

- Extending homomorphisms to endomorphisms, work of Cameron, Nešetřil, Lockett, Mašulović, Dolinka.