Graph inverse semigroups

Dave G. Jones

Heriot-Watt University

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Graph inverse semigroups

- The graph inverse semigroups are the most natural generalisation of the polycyclic monoids.
- They are also a special case of semigroups constructed from suitable left cancellative categories.
- We will generalise some of the theory from the polycyclic monoids and take them further.
- Introduce a new result inspired by certain graph algebras.

Categories

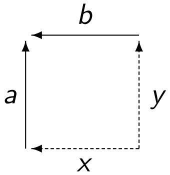
Throughout this talk categories are 'arrows-only'. The elements of a category C are called *arrows* and the set of identities of C is denoted by C_0. An arrow a is an *isomorphism* if there is a necessarily unique arrow a^{-1} such that $a^{-1}a$ and aa^{-1} are identities.

$$d(a) \bullet \xrightarrow{\quad a \quad} \bullet r(a)$$

A category in which every arrow is invertible is called a *groupoid*. We denote the subset of invertible elements of C by $G(C)$. This forms a groupoid. If $G(C) = C_0$ then we shall say that the groupoid of invertible elements is *trivial*.

Construction

Graph inverse semigroups are constructed as a special case of a general procedure for constructing inverse semigroups from left cancellative categories which has its origins in the work of Leech. The left cancellative categories required the additional condition that any pair of arrows with a common range that can be completed to a commutative square have a pullback. We call these *Leech categories*.



Equivalence

With each Leech category C, we may associate an inverse semigroup $\mathbf{S}(C)$ as follows. Put

$$U = \{(a, b) \in C \times C : \mathbf{d}(a) = \mathbf{d}(b)\}.$$

Define a relation $\sim$ on U as follows

$$(a, b) \sim (a', b') \Leftrightarrow (a, b) = (a', b')u$$

for some isomorphism $u \in C$. This is an equivalence relation on U and we denote the equivalence class containing (a, b) by $[a, b]$.

Product

The product $[a, b][c, d]$ is defined as follows: if there are no elements x and y such that $bx = cy$ then the product is defined to be zero; if such elements exist choose such a pair that is a pullback. The product is then defined to be $[ax, dy]$.

$$\begin{tikzcd}
 & & \bullet \\
 & \bullet & \bullet \arrow[l, "c"'] \arrow[u, "d"'] \\
\bullet & \bullet \arrow[l, "a"'] \arrow[u, "b"] & \bullet \arrow[l, dashed, "x"] \arrow[u, dashed, "y"']
\end{tikzcd}$$

Result

Define $\mathbf{S}(C)$ to be the set of equivalence classes together with an additional element that plays the role of zero.

Theorem

Let C be a Leech category. Then $\mathbf{S}(C)$ is an inverse semigroup with zero.

Structural Properties

Lemma

Let C be a Leech category. Then in the inverse semigroup $\mathbf{S}(C)$, we have the following:

1. The semigroup $\mathbf{S}(C)$ is E^*-unitary if and only if the Leech category C is right cancellative.
2. The inverse semigroup $\mathbf{S}(C)$ is 0-bisimple if and only if C is equivalent to a monoid.

Graphs

A *directed graph* can have loops and there can be multiple edges between any two vertices. The *free category* G^* generated by the directed graph G is the set of all paths equipped with concatenation as the partial binary operation.

A graph with one vertex and n loops is called the *n-rose* or *bouquet of n circles*. If the graph is an n-rose then its free category is simply the free monoid on n generators. This is why the polycyclic monoids are a special case of the graph inverse semigroups.

Graph inverse semigroups

When a Leech category has a trivial groupoid of invertible elements the equivalence class $[a, b]$ is just the singleton set $\{(a, b)\}$. It is convenient in this case to denote $[a, b]$ by ab^{-1}, which is to be understood to be just a notation.
The free category of a graph G is such a category. The multiplication takes the following form:

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some string } z \\ x(vz)^{-1} & \text{if } y = uz \text{ for some string } z \\ 0 & \text{otherwise} \end{cases}$$

The *graph inverse semigroup* is the special case of $\mathbf{S}(C)$ when C is the free category associated with a graph. For a graph G we denote the associated graph inverse semigroup by P_G.

Basic Properties

- Finite if and only if the graph is a finite tree.
- $(xy^{-1})^{-1} = yx^{-1}$.
- P_G is always combinatorial.
- All idempotent take the form xx^{-1}.
- The natural partial order: $xy^{-1} \leq uv^{-1} \Leftrightarrow x = up$ and $y = vp$.
- Only 0-bisimple in the special case of the polycyclic monoids.

Congruence free

The *strongly connected components* of a graph are the maximal strongly connected subgraphs.
In the worldwide bestseller *Inverse Semigroups* Lawson proved that the ploycyclic monoids are congruence free. This result generalises in the following way:

Theorem (1)

The P_G is congruence free if and only if the graph is strongly connected and the in-degree of each vertex is strictly greater than one.

Rees congruences

We say $u \in G_0$ is a *bridging vertex* if the in-degree of u is strictly greater than one and exactly one in-edge is from the same strongly connected component as u. A vertex u is *degenerate* if it has no cycles on it. This is equivalent to saying that the strongly connected component containing u is just a vertex with no edges.

Theorem (2)

Let G be a graph with no degenerate vertices and where the in-degree of each vertex is strictly greater than one. All congruences on P_G are Rees congruences if and only if there are no bridging vertices.

Wide inverse subsemigroups

A relation $\sim$ on a category C is called a *right congruence* if $x \sim y$ implies that $\mathbf{d}(x) = \mathbf{d}(y)$ and whenever xz, yz are defined we have that $xz \sim yz$.
A subsemigroup is called *wide* if it contains all the idempotents of the original semigroup.

Theorem (3)

Let G be a graph. Then the wide inverse subsemigroups of P_G are in bijective correspondence with the right congruences on G^*.

This is a generalisation of a result by Meakin and Sapir of the polycyclic monoids.

The gauge inverse semigroup

We define a relation $\sim$ on G^* by $a \sim b$ if $\mathbf{d}(a) = \mathbf{d}(b)$ and $|a| = |b|$. It is clear that $\sim$ is a right congruence and therefore there is an associated wide inverse subsemigroup of P_G which we will denote by Q_G. Explicitly:

$$Q_G = \{xy^{-1} \in P_G : |x| = |y|\}.$$

We call this the *gauge inverse semigroup*. The analogous structure plays an important role in certain algebras associated with graphs. The result on the next slide was inspired by such work.

The adjacency matrix of a graph

A subset I of S is said to be *co-finite* if it has finite complement. The *adjacency matrix* A of a finite graph G of n vertices is the $n \times n$ matrix where the a_{ij} entry is the number of edges from vertex i to vertex j. A matrix is said to be *aperiodic* if for some $m \in \mathbb{N}$ all the entries of A^m are non-zero.

Theorem (4)

Let G be strongly connected. The adjacency matrix is aperiodic if and only if all the ideals of Q_G are co-finite.

Finally

- Generalise more results about the polycyclic monoids.
- P_G is an unambiguous semigroup and we look to generalise the work further.
- Directed graphs give rise to Cuntz-Krieger C^*-algebras. There is a connection between these algebras and a certain completion of P_G.
- Translate the work on the associated algebras into results on P_G and the completion.

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- Translate the work on the associated algebras into results on P_G and the completion.

Thank you.

Structural Properties (1/2)

Lemma

Let C be a Leech category. Then in the inverse semigroup $S = \mathbf{S}(C)$, we have the following:

1. The semigroup $\mathbf{S}(C)$ is E^*-unitary if and only if the Leech category C is right cancellative.
2. The semigroup $\mathbf{S}(C)$ is combinatorial if and only if the invertible elements in each local monoid of C are identities.
3. Each $\mathcal{D}$-class of $\mathbf{S}(C)$ contains a unique maximal idempotent if and only if the only invertible elements are in the local monoids of C.

Structural Properties (2/2)

4. The groupoid of invertible elements in C is trivial if and only if $\mathbf{S}(C)$ is combinatorial and each $\mathcal{D}$-contains exactly one maximal idempotent.
5. The semigroup $\mathbf{S}(C)$ is unambiguous if and only if the category C is right rigid.
6. The inverse semigroup $\mathbf{S}(C)$ is completely semisimple if and only if for all identities e and f whenever eCf contains an isomorphism then every element of eCf is an isomorphism.
7. The inverse semigroup $\mathbf{S}(C)$ is 0-bisimple if and only if C is equivalent to a monoid.