Graphs to semigroups

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Graphs

- A graph consists of vertices and edges
- It can be directed or undirected
- It can contain loops
- Usually one considers only finite graphs

Algebra

- An algebra consists of a set A and one or more operations
- An operation is a function whose one or more arguments are from A and whose value is in A
- For example, addition or multiplication

Groupoid

- In this talk, every algebra has exactly one operation, and this operation is binary
- Usually the operation is called multiplication
- An algebra with one binary operation is called a groupoid

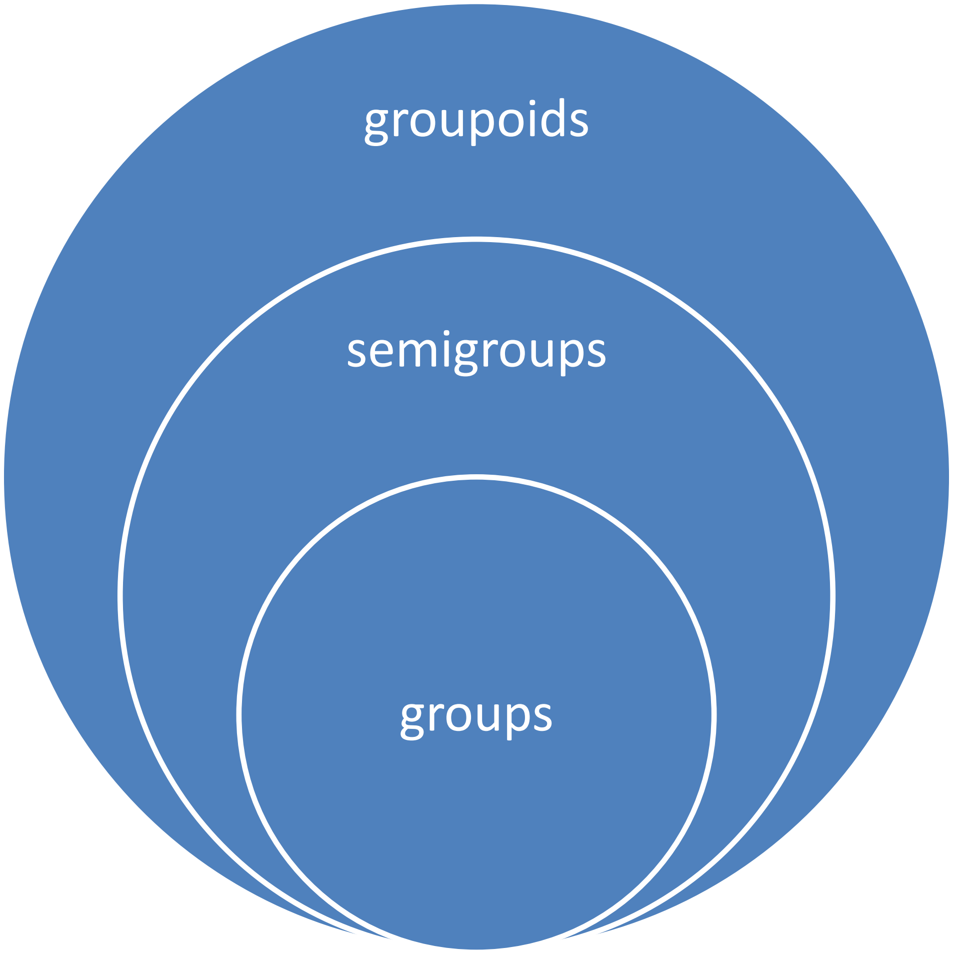
Semigroup

- A semigroup is a groupoid whose operation is associative
- The operation is called associative if $(ab)c=a(bc)$

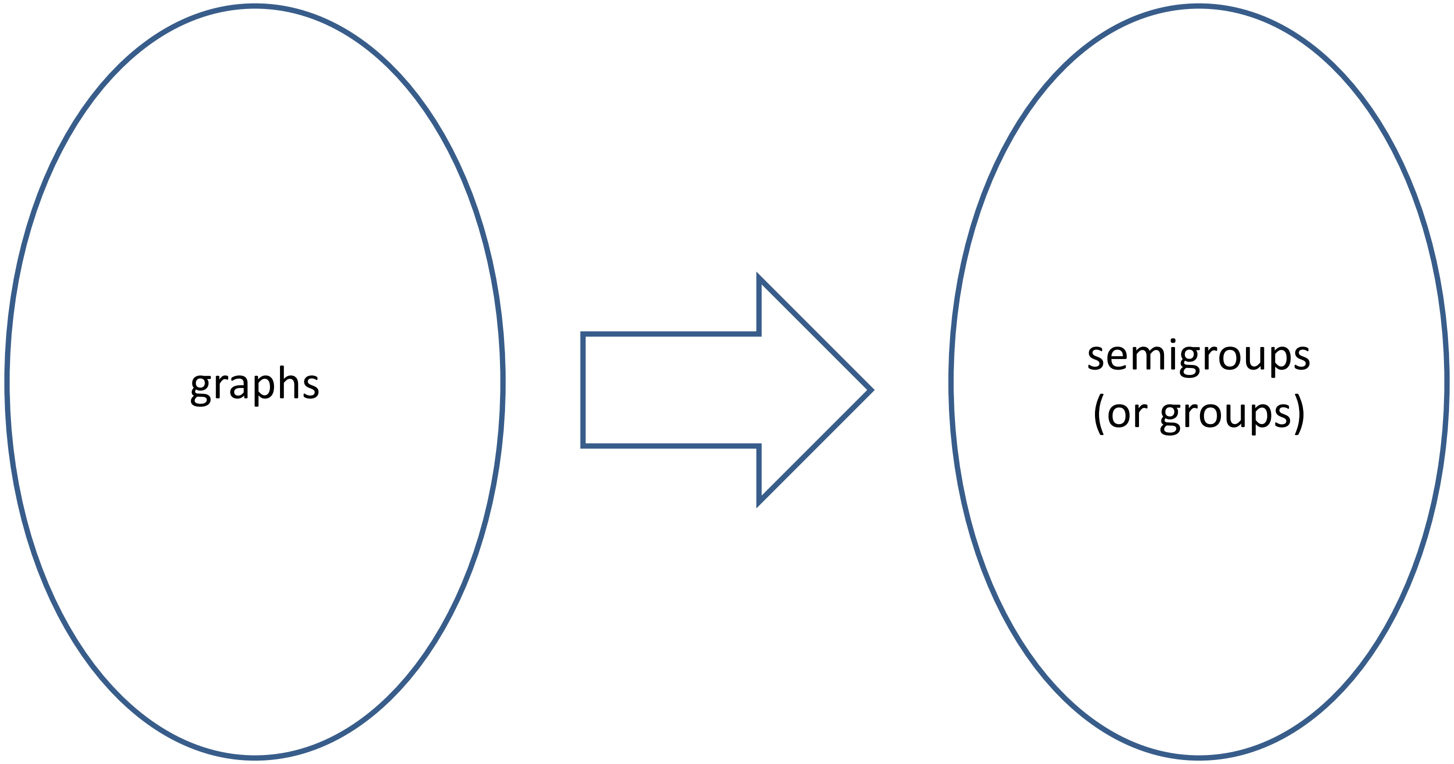
Group

- A group is a semigroup in which there is a neutral element and every element has an inverse
- Say, the neutral element is 1
- Then 1g=g1=g for every g
- And for every g there is g^{-1} such that
  $g\, g^{-1} = g^{-1} g=1$

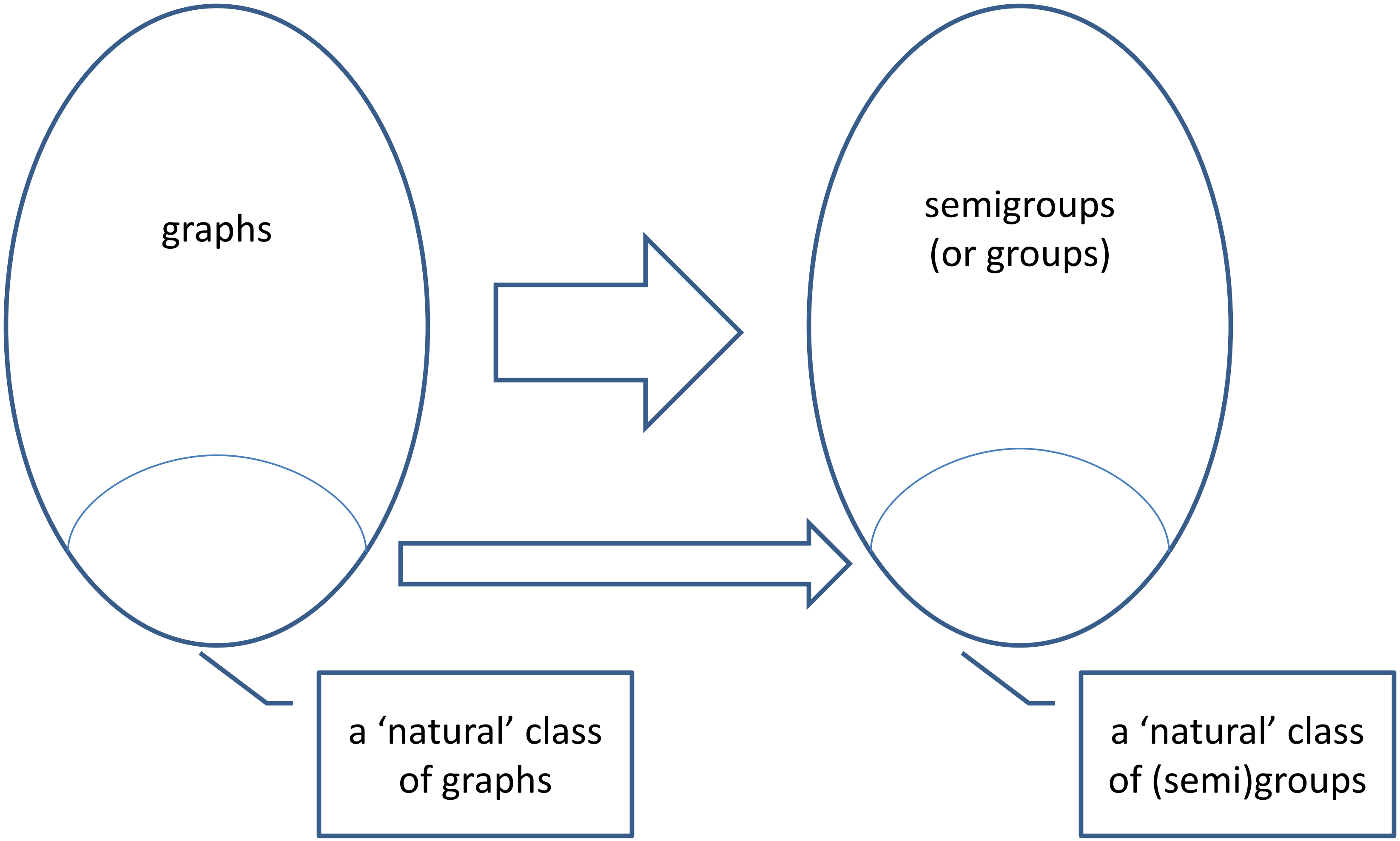
Types of algebras we consider



A correspondence



A ‘natural’ correspondence



A minor of a graph

- Graphs are assumed to be undirected.
- A graph H is called a minor of a graph G if H is isomorphic to a graph that can be obtained by edge contractions from a subgraph of G.
- In other words, delete some edges (and, perhaps, isolated vertices) in G, and then contract some paths.

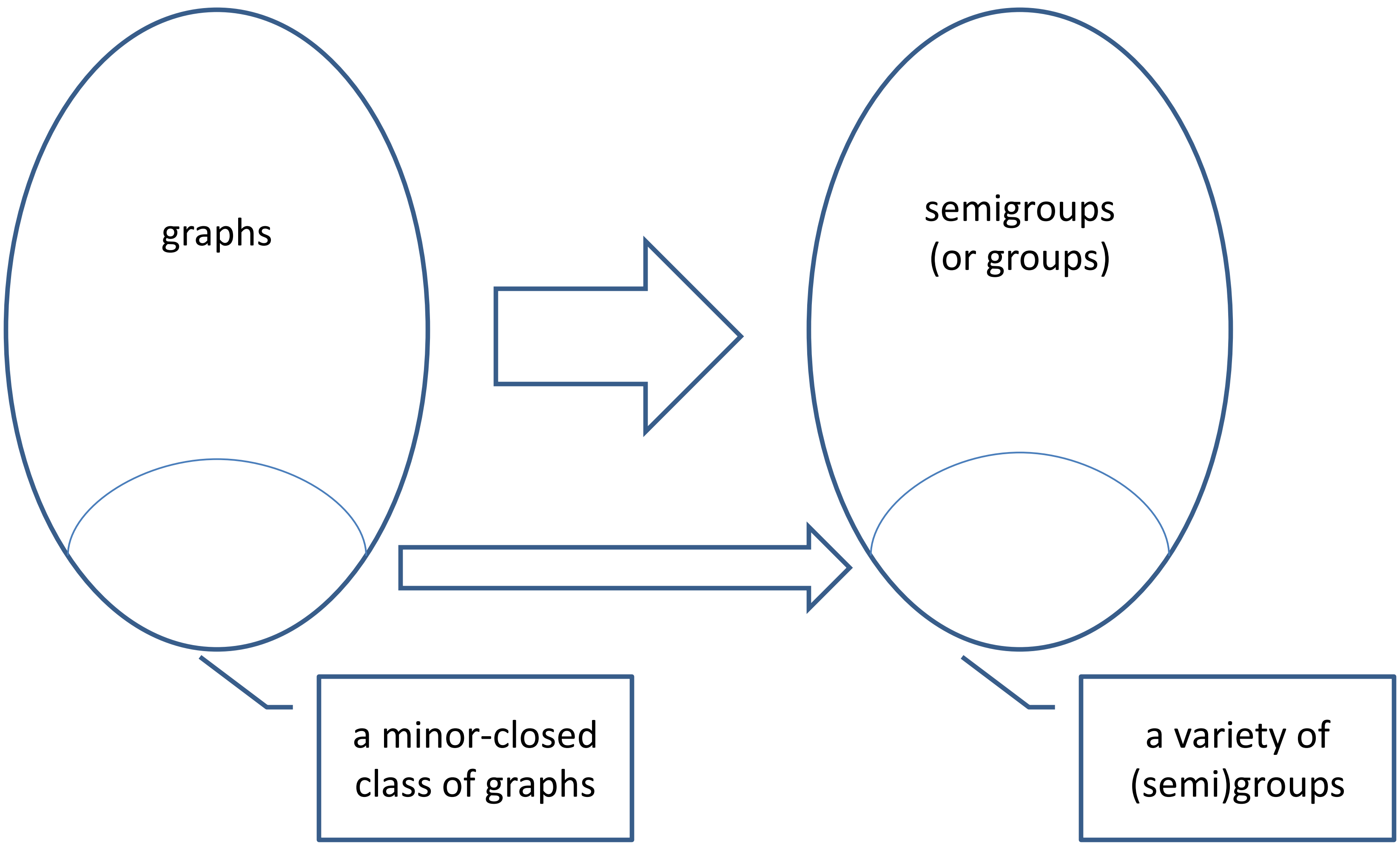
Minor-closed classes of graphs

- A minor-closed class of graphs is a class of graphs that is closed under taking minors.
- Minor-closed classes of graphs are described by the famous theorem of Neil Robertson and Paul D. Seymour.

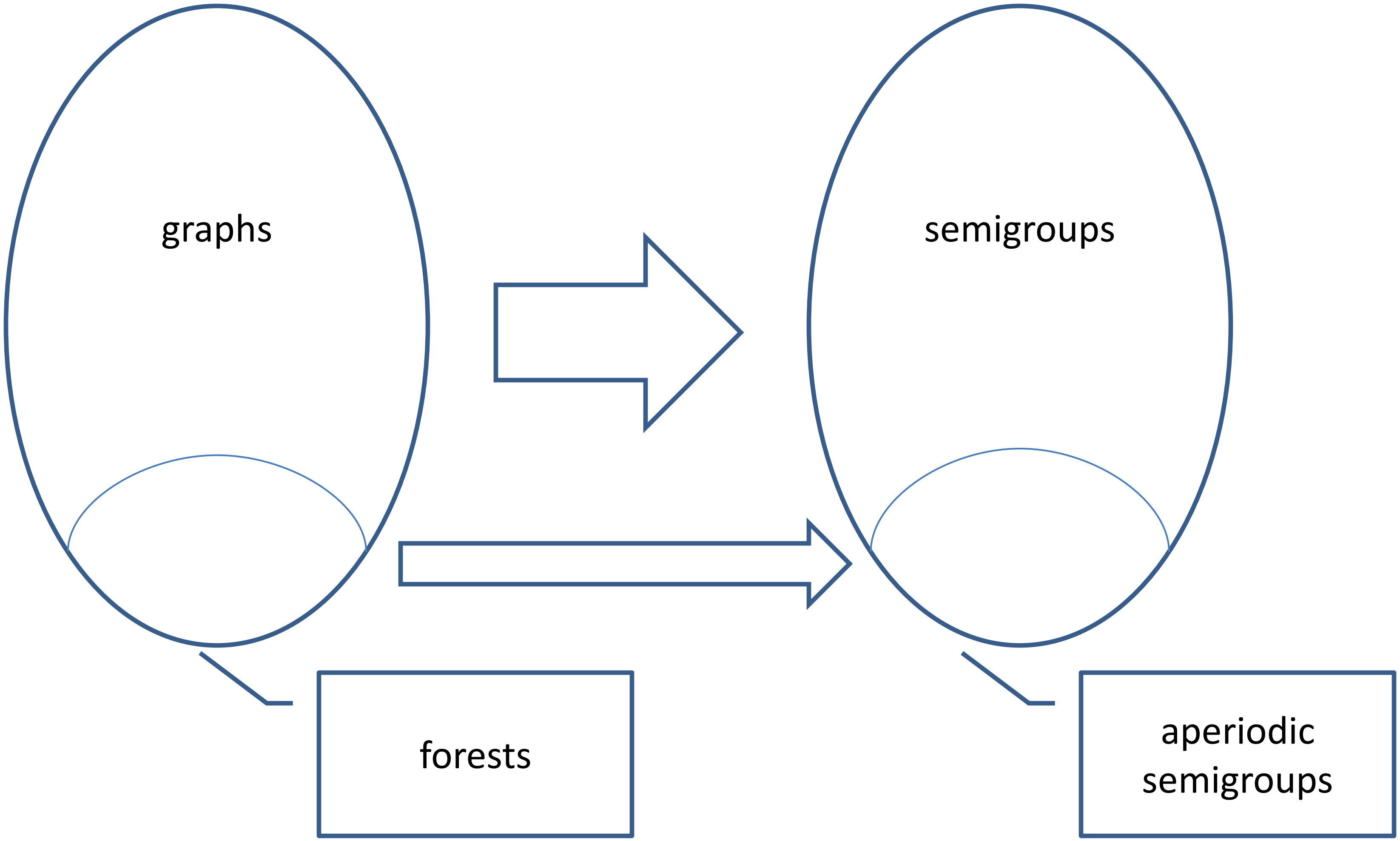
A variety of algebras

- A variety (pseudovariety) is a class of algebras which is closed with under taking subalgebras, factor-algebras and direct products (direct products of two algebras).
- Varieties are described by the famous theorem of Garrett Birkhoff, also known as the HSP theorem.
- Pseudovarieties also have several useful descriptions

For example (if it was possible)



For example (it is possible, in a way)



Graphs-to-algebra constructions
(in no particular order)

- Graph algebras
- Graph semigroups
- Graph groups
- Path semigroups
- Endomorphism semigroups
- Commutative graph semigroups
- Inverse graph semigroups
- Jackson-Volkov semigroups

Endomorphisms of graphs

- It is usually assumed that we consider undirected graphs
- An endomorphism of a graph is a transformation on the set of vertices which maps each pair of adjacent vertices to adjacent vertices
- If you have not decided yet, you need to decide if your graph has or does not have loops.

Endomorphism semigroups

- Endomorphisms of a graph form a semigroup
- Endomorphism semigroups of graphs are a standard object of research, like semigroups of endomorphisms of any other objects in discrete mathematics

Endomorphism semigroups

- Positive:
  - All endomorphism semigroups are finite, which is convenient
- Negative:
  - It’s impossible to reconstruct a graph from its endomorphism semigroup, even approximately

Graph algebras

- The word ‘algebra’ here stands for ‘groupoid’
- We introduce a binary operation on the set of vertices (with an added 0)
- The definition of multiplication uv is:
    - uv=u if there is an edge from u to v
    - uv=0 otherwise

Graph algebras

- Graph algebras are used to build ‘awkward’ examples of algebras in algebraic research
- For example, ‘usually’ the variety generated by a graph algebra cannot be defined by finitely many identities.

Graph algebras

- Positive:
  - The graph algebra of a graph reflect the structure of the graph
- Negative:
  - Algebraic properties of graph algebras are inconvenient

Graph semigroups

- Graphs are assumed to be undirected.
- We generate a semigroup whose set of generators is the set of vertices
- The defining relations are uv=vu for each pair of adjacent vertices u,v.
- (Or, in old papers, uv=vu if u and v are **not** adjacent)

Graph semigroups

- Graph semigroups are an area of research within semigroup theory.
  - They are a generalisation of free semigroups
  - Many word problem results have been proved

Graph semigroups

- Positive:
  - The graph semigroup of a graph reflect the structure of the graph
- Negative:
  - Graph semigroups are infinite
  - Graph semigroups don’t seem sufficiently versatile
  - This is why they are also known as free partially commutative semigroups

Graph groups

- This is a ‘group version’ of graph semigroups
- Graphs are assumed to be undirected.
- We generate a group whose set of generators is the set of vertices
- The defining relations are uv=vu for each pair of adjacent vertices u,v.

Graph groups

- Graph groups are used to build interesting examples of groups
- (which are similar to braid groups)

Graph groups

- Positive:
  - The graph group of a graph reflect the structure of the graph
- Negative:
  - Graph groups are infinite
  - Graph groups don’t seem sufficiently versatile

Graph (semi)groups

- Negative
  - A smaller or larger (semi)group can correspond to a smaller or larger graph
  - Indeed,
    - Adding (removing) vertices makes the semigroup larger (smaller)
    - Adding (removing) edges makes the semigroup smaller (larger)

Commutative graph semigroups

- Graphs are assumed to be undirected.
- We generate a commutative semigroup whose set of generators is the set of vertices
- The defining relations are $u=v_1+...+v_k$ for each vertex u, where $v_1,...,v_k$ are all vertices adjacent to u.

Commutative graph semigroups

- This is a new interesting object in algebra
- Commutative graph semigroups have been applied to the study of rings and modules (this is two types of algebras)

Commutative graph semigroups

- Positive:
  - The commutative graph semigroup of a graph reflect the structure of the graph (to some extent)
  - It’s a convenient object to work with
- Negative:
  - Commutative graph semigroups are infinite
    - Although maybe we can use coefficients reduced modulo 2?
  - Commutative graph semigroups don’t seem sufficiently versatile
    - They all are commutative

Inverse graph semigroups

- There is a construction which puts an inverse semigroup in correspondence with a graph
- I don’t know much about this construction
- Inverse graph semigroups are used to study C*-algebras and category-theory generalisations of graphs

Path semigroups

- We generate a semigroup whose set of generators is the set of vertices and edges (with an added 0)
- The defining relations are
  - u(u,v)=(u,v)v=(u,v), where (u,v) is an edge from u to v
  - uv=0, if u and v are two distinct vertices

Path semigroups

- Path semigroups are used in algebra, for example, in group theory and in the study of C*-algebras.

Path semigroups

- Positive:
  - The path semigroup of a graph reflect the structure of the graph
  - We can consider finite path semigroups
- Negative:
  - Path semigroups don’t seem sufficiently versatile
    - ‘Most’ products are 0

Jackson-Volkov semigroups

- We consider a semigroup whose elements are pairs of vertices (with an added 0)
- We define the product (u,v)(x,y)
  - It is (u,y) if there is an edge (v,x)
  - It is 0, otherwise
- Also, we add a unary operation of reversing a pair

Jackson-Volkov semigroups

- There is a correspondence between universal Horn classes of graphs and varieties of Jackson-Volkov semigroups

Jackson-Volkov semigroups

- Positive:
  - The Jackson-Volkov semigroup of a graph reflects the structure of the graph
  - Jackson-Volkov semigroups are finite
- Negative:
  - Jackson-Volkov semigroups don’t seem sufficiently versatile
    - They are 0-simple

My construction

- Consider the semigroup generated by the following transformations on the set of vertices
  - For each edge (u,v), a transformation mapping u to v and leaving everything else where it is
- Let us call this semigroup the transformation graph semigroup, and denote by TS(G), where G is the graph

Transformation graph semigroups

- Positive:
  - The transformation semigroup of a graph reflects the structure of the graph
  - Transformation graph semigroups are finite
  - Transformation graph semigroups are sufficiently different from one another
    - For example, for any identity, one can present a graph whose transformation semigroup does not satisfy the identity.

Transformation graph semigroups

- Properties of a graph are naturally reflected in the properties of its transformation semigroup

Path length

- There is a path of length k in G if and only if there is an element s of the index k in TS(G) (that is, $s, s^2, \ldots, s^k$ are pairwise distinct)
- There is a cycle of length k in G if and only if there is a group element s of the order k-1 in TS(G)
  (that is, $s=s^k$)

Corollaries

- G is a forest if and only if TS(G) is aperiodic.
- G is bipartite if and only if every group element of TS(G) has an odd order.
- G is a disjoint union of stars if and only if the identity $x^3=x^4$ holds in TS(G)

Connectivity

- G is connected if and only if TS(G) is subdirectly indecomposable
- Therefore, G is a tree if and only if TS(G) is aperiodic and subdirectly indecomposable.

Towards minors

- (Conjecture) Every transformation graph subsemigroup of TS(G) corresponds to a minor of G, and vice versa, every minor of G corresponds to a transformation graph subsemigroup of TS(G).
- By ‘transformation graph subsemigroup’ we mean a subsemigroup which is isomorphic to a transformation graph semigroup

Abstract characterisation

- At the moment, I am trying to develop an abstract characterisation of semigroups isomorphic to transformation graph semigroups.