# Approaches to tiling semigroups

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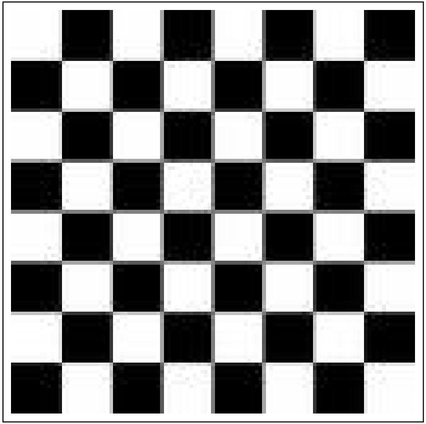
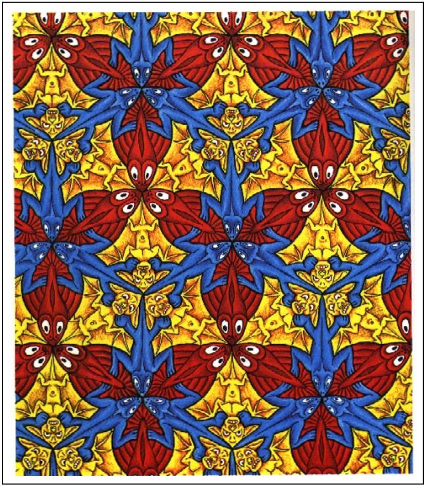
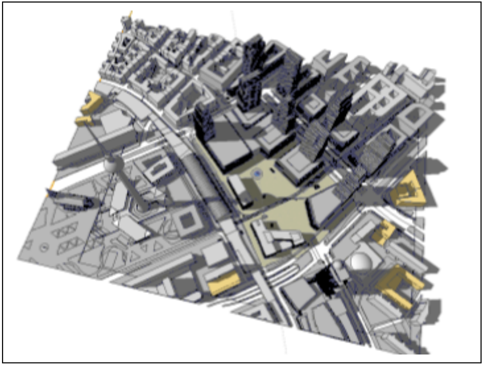
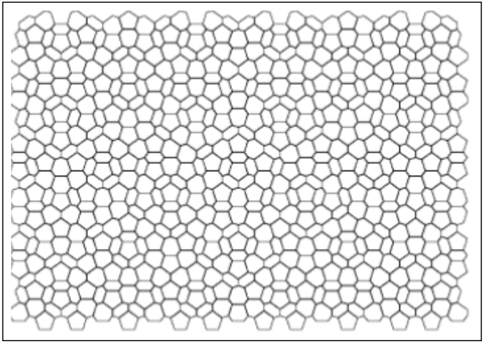
# Periodic tilings





# Non-periodic tilings





Ulrich Kortenkamp: *Paving the Alexanderplatz*

# Tilings of $\mathbb{R}^n$

A *tile* in $\mathbb{R}^n$ is a connected bounded subset $t$ that is the closure of its interior, with a colour assigned from a set $\Sigma$.

A *tiling* $\mathcal{T}$ of $\mathbb{R}^n$ is a union of tiles meeting only at their boundaries.

We always assume there are only finitely many different types of tiles: type $\equiv$ equivalence up to $\Sigma$-preserving translation in $\mathbb{R}^n$.

One tile type and one colour give a vanilla tiling of $\mathbb{R}^n$.

A marked pattern in $\mathcal{T}$ is a finite connected collection of tiles in $\mathcal{T}$, with two chosen distinguished tiles, the **in-tile** and the **out** $-$ **tile** (possibly equal) – up to equivalence under translation.

# History

- Johannes Kellendonk (1997): intro tiling semigroups
- Kellendonk & Mark Lawson (2000): general theory
- Yongwen Zhu (2002): algebraic properties
- Lawson (2004): specific theory for one-dimensional case
- Alan Forrest, John Hunton & Kellendonk (1999-2008): cohomological invariants
- Don McAlister & Filipa Soares (2005-2010)
- Erzsi Dombi & NDG (2007-10)

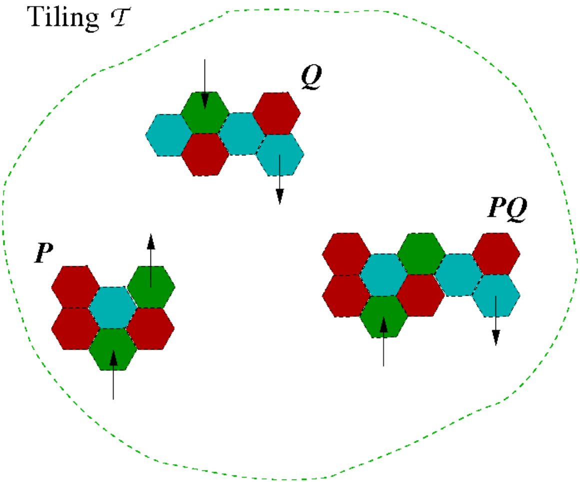
# The tiling semigroup

- tiling $\mathcal{T}$ has tiling semigroup $S(\mathcal{T})$, with $0 \in S(\mathcal{T})$,
- non-zero elements are marked patterns,
- multiply $P, Q$ by matching out-tile of $P$ with in-tile of $Q$: if resultant overlap matches, and then if $P \cup Q$ is a pattern, $PQ = P \cup Q$ marked at the in-tile of $P$ and the out-tile of $Q$. Non-matching $\implies PQ = 0$.

Three possible reasons for $PQ = 0$:

- in- and out- tiles don't match,
- rest of overlap doesn't match,
- $P \cup Q$ isn't a pattern in $\mathcal{T}$

# Multiplication in $S(\mathcal{T})$



# Basic properties of tiling semigroups

- idempotent $\iff$ in-tile = out-tile
- tiling semigroups are inverse: get $P^{-1}$ by switching the in- and out- tiles
- tiling semigroups are combinatorial, completely semisimple, $E^*$–unitary inverse semigroups.
- natural partial order is reverse inclusion of marked patterns:

$$P \leq Q \iff P \supseteq Q \text{ as marked patterns}$$

so big patterns are below small ones in the natural partial order.

# One dimensional tilings

Fix a finite alphabet $A = \{a_1, \ldots, a_n\}$ ($n \geq 2$ most of the time). We can identify a *one dimensional tiling* as a bi-infinite string over $A$:

$$\mathcal{T} = \cdots a_{i_{-2}} a_{i_{-1}} a_{i_0} a_{i_1} a_{i_2} \cdots$$

- Language $L(\mathcal{T})$ is the set of finite substrings of $\mathcal{T}$
- $L(\mathcal{T})$ is *factorial* – closed under substrings
- A *marked pattern* is now a word in $L(\mathcal{T})$ with distinguished in- and out- letters $\grave{a}$ and $\acute{b}$

# The free monogenic inverse monoid

- denote by $\mathrm{FIM}_1$
- tiling semigroup of 1-diml vanilla tiling (with 0 removed)
- elements are marked strings on a single letter $t$
- multiply by matching out- and in- markers and superposing strings
- strings always match, and $\check{t} = 1$
- gap between in- and out- letter is a hom to $\mathbb{Z}$
- Give coords in $\mathbb{Z}^3$: $t_p t_{p+1} \cdots t_{-1} \grave{t}_0 t_1 \cdots \acute{t}_r \cdots t_q \mapsto (p, q, r)$

# A presentation for $S(\mathcal{T})$

Let $\mathcal{T}$ be a one-dimensional tiling with alphabet $A = \{a_1, \ldots, a_n\}$.

**Theorem (McAlister-Soares 2006)**

*$S(\mathcal{T})$ is generated by the single-tile idempotents $\check{a}_i$ and the two-tile patterns $t_{ij} = \grave{a}_i \acute{a}_j$, with defining relations*

$$\check{a}_i{}^2 = \check{a}_i, \check{a}_i \check{a}_j = 0,$$

$$\check{a}_i t_{ij} = t_{ij} = t_{ij} \check{a}_j$$

$$t_{ij}^{-1} t_{ik} = 0 \text{ if } j \neq k, t_{ij} t_{kj}^{-1} = 0 \text{ if } i \neq k$$

$$w = 0 \text{ if } w \text{ has underlying word} \notin L(\mathcal{T})$$

# Finite presentability

### Corollary

*$S(\mathcal{T})$ is a finitely presented inverse semigroup iff $L(\mathcal{T})$ is a locally testable language.*

Locally testable: determine membership by substrings of fixed length. Factorial and locally testable iff membership determined by finitely many forbidden subwords.
More on finite presentability in dimension $> 1$ later.

# $F^*$–inverse

1-diml tiling semigroups are $F^*$–inverse: each non-zero element is below a unique maximal element in the natural partial order:
$u \leq \widehat{u}$ where $\widehat{u}$ is the smallest substring of $\pi$ carrying the in- and out- tiles of the marked string $u$.
$n$-diml tiling semigroups ($n > 1$) need not be $F^*$–inverse . . .
. . . but more of this later.

# Periodic tilings

For detailed structure, simplify to *periodic* $\mathcal{T}$: repeat fixed finite string *ad bi-infinitum* . . .

$$\cdots abcababcababcababcab \cdots$$

**Theorem (Dombi-NDG 2009)**

*The tiling semigroup of a one-dimensional periodic tiling with period of length $m$ embeds into $\mathcal{P}(\mathbb{Z}_m) \rtimes_0 \mathsf{FIM}_1$, and the subsemigroups that arise are completely determined.*

# Embeds where?

- $\mathcal{P}(\mathbb{Z}_m)$ – power set of subsets of $\mathbb{Z}_m$
- $\mathrm{FIM}_1$ – free monogenic inverse monoid
- $\mathrm{FIM}_1 \to \mathbb{Z} \to \mathbb{Z}_m$ so $\mathrm{FIM}_1$ acts on $\mathcal{P}(\mathbb{Z}_m)$ by translation
- $\mathcal{P}(\mathbb{Z}_m) \rtimes_0 \mathrm{FIM}_1$ is the semidirect product of monoids, with $(\emptyset, w) = 0$ for all $w \in \mathrm{FIM}_1$.

# Sketch proof of the theorem

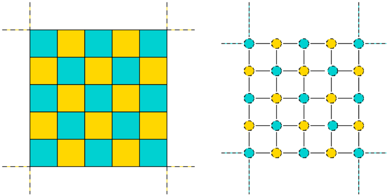
$\mathcal{T}$ periodic with alphabet $A = \{t_1, t_2, \ldots, t_n\}$ and period $p_0 p_1 \cdots p_{m-1}$. Let $\pi$ be a marked string in $\mathcal{T}$ of length $\ell$.
Find coords $\tau(\pi) \in \mathbb{Z}^3$ just by counting (ignore spelling!)
If $\ell \geq m$, the in-letter is a unique $p_j$ and we set $\Omega(\pi) = \{j\}$, but if $\ell < m$ then the in-letter may be $p_j$ with $j$ in some finite subset $\Omega(\pi)$.

**Theorem**

*$S(\mathcal{T})$ embedded in $\mathcal{P}(\mathbb{Z}_m) \rtimes_0 \mathrm{FIM}_1$ by $\pi \mapsto (\Omega(\pi), \tau(\pi))$.*

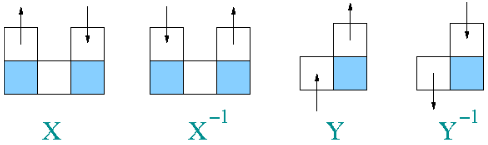
# Hypercubic tilings of $\mathbb{R}^n$

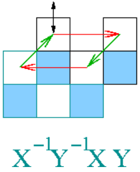
- McAlister-Soares 2009
- tiling $\mathbb{R}^n$ by (coloured) regular cubic lattice
- dual graph is the Cayley graph of $\mathbb{Z}^n$ with coloured vertices
- tiling is replaced by colouring map $\chi : \mathbb{Z}^n \to \Sigma$



# Commutators are idempotents:

In any tiling semigroup $S(\mathcal{T})$, $u^{-1}v^{-1}uv$ is an idempotent: distance between in-tile and out-tile must be 0.



# High dimensional, but vanilla

For the $n$–diml vanilla hypercubic tiling $\mathcal{V}$:

- all patterns have non-zero product, so 0 is removable
- Set
  $M = \mathrm{Inv}\langle a, b : (u^{-1}v^{-1}uv)^2 = u^{-1}v^{-1}uv (u, v \in \{a^{\pm}, b^{\pm}\}^{+}\rangle$
- Margolis-Meakin (1989): $M$ is the universal $E$–unitary inverse monoid with max group image $\mathbb{Z}^2$
- So have $\phi : M \to S(\mathcal{V})$

**Theorem (McAlister-Soares 2009)**

$\ker\phi$ *is not finitely generated as a congruence on* $M$*.*

# LF Y-HNN

Locally full Yamamura HNN-extensions of semilattices with zero:

- $E$ a semilattice, $0 \in E$: $(e)^{\downarrow} = \{x \in E : x \leq e\}$,
- Family of isomorphisms $\phi_i : (e_i)^{\downarrow} \to (f_i)^{\downarrow}$,
- $S = \mathrm{Inv}_0\langle E, t_i : t_i t_i^{-1} = e_i, t_i^{-1} t_i = f_i, t_i^{-1} x t_i = x\phi_i \ (x \leq e_i)\rangle$

**Theorem (Yamamura 1999)**

*$E(S) = E$, every subgroup of $S$ is free, and $S$ is (strongly) $F^*$–inverse.*

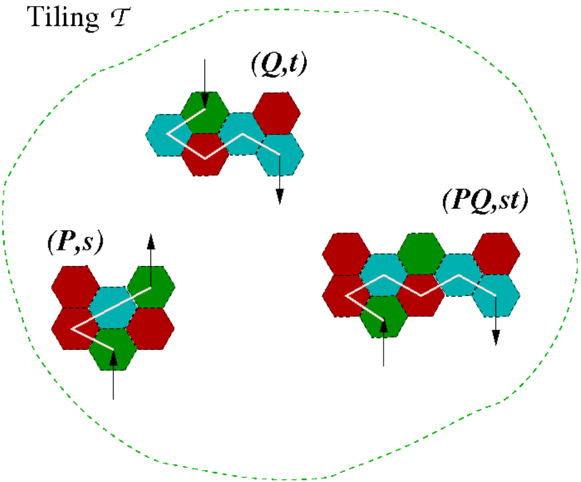
# The path extension

For an $n$-diml tiling $\mathcal{T}$, non-zero elements of the path extension $\Pi(\mathcal{T})$ are pairs $(P, u)$ where

- $P \in S(\mathcal{T})$ is a marked pattern,
- $u$ is a reduced path in the dual graph of $\mathcal{T}$ from the in-tile to the out-tile of $P$

Multiply componentwise: $(P, u)(Q, v) = (PQ, uv)$, where $uv$ is the free reduction of the concatenation of $u$ and $v$ (ie we work in the fundamental groupoid of the dual graph).

# Multiplication in $\Pi(\mathcal{T})$



# Path extension covers...

For any $n$-diml tiling $\mathcal{T}$:

**Theorem (Dombi-NDG 2010?)**

*The path extension is a strongly $F^*$–inverse cover of $S(\mathcal{T})$ and is isomorphic to a locally full Yamamura HNN extension of $E(S(\mathcal{T}))$. (If $n = 1$ the covering map is an isomorphism.)*

# Y-HNN ingredients

- $E = E(S(\mathcal{T})) = E(\Pi(\mathcal{T}))$,
- Idempotents $e_i$ are all possible two-tile patterns with a fixed choice of marked tile,
- Idempotents $f_i$ choose the other tile,
- Isom $\phi : E_i \to F_i$ switches the marking.