Structure Theorems for Proper Restriction Semigroups

Claire Cornock
Supervised by Victoria Gould

University of York

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Definition

An element $a' \in S$ is an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If each element of S has exactly one inverse in S, then S is an *inverse semigroup*.

Definition

For $a, b \in S$,

$$a \,\mathcal{R}\, b \Leftrightarrow a = bt \text{ and } b = as \text{ for some } s, t \in S$$

and

$$\begin{aligned} a \,\sigma\, b &\Leftrightarrow ea = eb \text{ for some } e \in E(S) \\ &\Leftrightarrow af = bf \text{ for some } f \in E(S). \end{aligned}$$

E-unitary and Proper Inverse Semigroups

Definition

An inverse semigroup is *proper* if and only if $\mathcal{R} \cap \sigma = \iota$, i.e.

$$a \, \mathcal{R} \, b \text{ and } a \, \sigma \, b \Leftrightarrow a = b.$$

Definition

An inverse semigroup S is *E-unitary* if for all $a \in S$ and all $e \in E(S)$, if $ae \in E(S)$, then $a \in E(S)$.

Proposition

Let S be an inverse semigroup. Then the following are equivalent:
i) S is E-unitary;
ii) S is proper;
iii) $\mathcal{L} \cap \sigma = \iota$.

McAlister's Covering Theorem

Definition

Let S be an inverse semigroup. An *E-unitary cover* of S is an E-unitary inverse semigroup U together with an onto morphism

$$\psi : U \to S$$

where ψ is idempotent separating.

McAlister's Covering Theorem

Every inverse semigroup has a E-unitary cover.

McAlister Triples

Definition

Let G be a group and let $(\mathcal{X}, \leq)$ be a partially ordered set where G acts on $\mathcal{X}$ by order automorphisms. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$.
Suppose that the following conditions are satisfied:
P1) $\mathcal{Y}$ is a semilattice under $\leq$;
P2) $G\mathcal{Y} = \mathcal{X}$;
P3) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$;
P4) For all $g \in G$, $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$.
Then $(G, \mathcal{X}, \mathcal{Y})$ is called a *McAlister triple*.

P-Semigroups

Definition

Let $(G, \mathcal{X}, \mathcal{Y})$ be a McAlister triple. The set

$$P(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G : g^{-1}A \in \mathcal{Y}\},$$

with the binary operation defined by

$$(A, g)(B, h) = (A \wedge gB, gh)$$

for $(A, g), (B, h) \in P(G, \mathcal{X}, \mathcal{Y})$, is called a *P-semigroup*.

McAlister's P-Theorem

Let P be a P-semigroup. Then P is an E-unitary inverse semigroup. Conversely, any E-unitary inverse semigroup is isomorphic to a P-semigroup.

Definition

Suppose S is a semigroup and E a set of idempotents of S. Let $a, b \in S$. Then $a \widetilde{\mathcal{R}}_E \, b$ if and only if for all $e \in E$,

$$ea = a \text{ if and only if } eb = b.$$

Definition

A semigroup S is *left restriction* (formally known as *weakly left E-ample*) if the following hold:

1) E is a subsemilattice of S;
2) Every element $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to an idempotent in E (idempotent denoted by a^+);
3) $\widetilde{\mathcal{R}}_E$ is a left congruence;
4) For all $a \in S$ and $e \in E$,

$$ae = (ae)^+a \text{ (the } \textit{left ample condition}\text{)}.$$

Proper Restriction and Weakly Ample Semigroups

Let S be a left restriction semigroup with distinguished semilattice E. Then for $a, b \in S$,

$$a \, \sigma_E \, b \Leftrightarrow ea = eb \text{ for some } e \in E.$$

Definition

A left restriction semigroup is *proper* if and only if $\widetilde{\mathcal{R}}_E \cap \sigma_E = \iota$.

A right restriction semigroup is *proper* if and only if $\widetilde{\mathcal{L}}_E \cap \sigma_E = \iota$.

Ample Semigroups

Definition

Let S be a semigroup and let $a, b \in S$. Then $a \mathcal{R}^* b$ if and only if for all $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb.$$

Proposition

Let $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ be the relations defined above on a semigroup S. Then

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E.$$

Ample Semigroups

Definition

A semigroup S is *left ample* (formally known as *left type A*) if the following hold:

1) $E(S)$ is a subsemilattice of S;
2) Every element $a \in S$ is $\mathcal{R}^*$-related to an idempotent in $E(S)$ (idempotent denoted by a^+);
3) For all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+ a.$$

Definition

A left ample semigroup is *proper* if and only if $\mathcal{R}^* \cap \sigma = \iota$.

A right ample semigroup is *proper* if and only if $\mathcal{L}^* \cap \sigma = \iota$.

Background Work: Structure Theorem for Proper Ample Semigroups

Suppose the following hold:

(1) $\mathcal{X}$ is a partially ordered set;

(2) $\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$;

(3) $\varepsilon \in \mathcal{X}$ such that $a \leq \varepsilon$ for all $a \in \mathcal{Y}$;

(4) T is a right cancellative monoid, which acts by order endomorphisms on the left of $\mathcal{X}$;

(5) $T\mathcal{Y}^i = \mathcal{X}$, where $\mathcal{Y}^i = \mathcal{Y} \cup \{i\}$;

(6) For $t \in T$, $\exists b \in \mathcal{Y}$ such that $b \leq t\varepsilon$;

(7) If $a, b \in \mathcal{Y}$, and $a \leq t\varepsilon$, then $a \wedge tb \in \mathcal{Y}$;

(8) If $a, b, c \in \mathcal{Y}$ and $a \leq t\varepsilon$ and $b \leq u\varepsilon$, then

$$(a \wedge tb) \wedge tuc = a \wedge t(b \wedge uc).$$

Background Work: Structure Theorem for Proper Ample Semigroups

Given $(T, \mathcal{X}, \mathcal{Y})$ as above, we define

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(a, t) \in \mathcal{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation

$$(a, t)(b, u) = (a \wedge t \cdot b, tu)$$

for $(a, t), (b, u) \in \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$.

The triple $(T, \mathcal{X}, \mathcal{Y})$ is called a *left admissible triple* and $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ an *M-semigroup*.

Theorem (Fountain)

An M-semigroup is proper left ample. Conversely, a proper left ample semigroup is isomorphic to an M-semigroup for some left admissible triple $(T, \mathcal{X}, \mathcal{Y})$.

Background Work: Structure Theorem for Proper Ample Semigroups

Let $(T, \mathcal{X}, \mathcal{Y})$ be a left admissible triple and $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ an M-semigroup. The triple $(T, \mathcal{X}, \mathcal{Y})$ is called an *admissible triple* if the following hold:

(A) There is a (unique) element $[a, t] \in \mathcal{Y}$ for every $(a, t) \in \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ such that $a \leq t \cdot [a, t]$ and $\forall c, d \in \mathcal{Y}$,

$$a \wedge tc = a \wedge td \Rightarrow [a, t] \wedge c = [a, t] \wedge d;$$

(B) For $e \in \mathcal{Y}$ and $a \in \mathcal{Y}$ with $a \leq t \cdot \varepsilon$,

$$a \wedge e = a \wedge t \cdot [e \wedge a, t];$$

(C) For $a, b \in \mathcal{Y}$ with $a, b \leq t \cdot \varepsilon$, $[a, t] = [b, t] \Rightarrow a = b$.

Theorem (Lawson)

Let S be a proper ample semigroup. Then $S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ for some admissible triple $(T, \mathcal{X}, \mathcal{Y})$. Conversely, every admissible triple gives rise to an M-semigroup, which is proper ample.

Background Work: Structure Theorem for Proper Left Restriction Semigroups

Suppose the following hold:

(1) $\mathcal{X}$ is a semilattice;

(2) $\mathcal{Y}$ is a subsemilattice of $\mathcal{X}$;

(3) $\varepsilon \in \mathcal{X}$ such that $a \leq \varepsilon$ for all $a \in \mathcal{Y}$;

(4) T is a monoid, which acts by morphisms on the left of $\mathcal{X}$;

(5) For all $t \in T$, there exists $a \in \mathcal{Y}$ such that $a \leq t \cdot \varepsilon$;

(6) For all $a, b \in \mathcal{Y}$ and all $t \in T$,

$$a \leq t \cdot \varepsilon \Rightarrow a \wedge t \cdot b \text{ lies in } \mathcal{Y}.$$

Given $(T, \mathcal{X}, \mathcal{Y})$ as above, we define

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(a, t) \in \mathcal{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation defined for $(a, t), (b, u) \in \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ by

$$(a, t)(b, u) = (a \wedge t \cdot b, tu).$$

Background Work: Structure Theorems for Proper Left Restriction and Weakly Left Ample Semigroups

Theorem (Branco, Gould, Gomes)

If T is an arbitrary monoid, a strong M-semigroup is a proper left restriction semigroup. Conversely, a proper left restriction semigroup is isomorphic to a strong M-semigroup.

Theorem (Gould, Gomes)

If T is a unipotent monoid, $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is a proper weakly left ample semigroup. Conversely, a proper weakly left ample semigroup is isomorphic to a strong M-semigroup where T is unipotent.

Structure Theorem for Proper Left Ample Semigroups

Theorem

If T is right cancellative, $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ is a proper left ample semigroup. Conversely, a proper left ample semigroup is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$, where T is right cancellative.

Theorem

A proper inverse semigroup is isomorphic to $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$, *where* T *is a group and* $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ *is a 'strong M-semigroup' with altered condition*

(5) *For every* $t \in T$, $\exists a \in \mathcal{Y}$ *such that* $a \leq t \cdot \varepsilon$ *and* $t^{-1} \cdot a \in \mathcal{Y}$

and

$$\mathcal{M}(T, \mathcal{X}, \mathcal{Y}) = \{(a, t) \in \mathcal{Y} \times T : a \leq t \cdot \varepsilon, t^{-1} \cdot a \in \mathcal{Y}\}.$$

Conversely, $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$, *with altered condition (5) and* T *a group, is a proper inverse semigroup.*

Structure Theorem for Proper Restriction Semigroups

Let $(T, \mathcal{X}, \mathcal{Y})$ be strong left M-triple and $\mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ a strong M-semigroup. The triple $(T, \mathcal{X}, \mathcal{Y})$ is called a *strong M-triple* if the following hold:

(A) There is a (unique) element $[a, t] \in \mathcal{Y}$ for every $(a, t) \in \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ such that $a \leq t \cdot [a, t]$ and $\forall f \in \mathcal{Y}$,

$$a \leq t \cdot f \Rightarrow [a, t] \leq f;$$

(B) For all $(a, t), (b, u), (x, y) \in \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$,

$$\forall e \in \mathcal{Y}, [a \leq t \cdot e \Leftrightarrow b \leq u \cdot e]$$

$$\Rightarrow$$

$$\forall f \in \mathcal{Y}, [a \wedge t \cdot x \leq ty \cdot f \Leftrightarrow b \wedge u \cdot x \leq uy \cdot f];$$

(C) For $e \in \mathcal{Y}$ and $a \in \mathcal{Y}$ with $a \leq t \cdot \varepsilon$,

$$a \wedge e = a \wedge t \cdot [e \wedge a, t];$$

(D) For $a, b \in \mathcal{Y}$ with $a, b \leq t \cdot \varepsilon$, $[a, t] = [b, t] \Rightarrow a = b$.

Structure Theorem for Proper Restriction Semigroups

Theorem

Let S be a proper restriction semigroup. Then $S \cong \mathcal{M}(T, \mathcal{X}, \mathcal{Y})$ for some strong M-triple. Conversely, every strong M-triple gives rise to a strong M-semigroup, which is proper restriction.

Motivation for a two-sided structure theorem

Definition (Fountain, Gomes, Gould)

A monoid T acts doubly on a semilattice $\mathcal{Y}$ with identity, if

(i) *T acts by morphisms on the left and right of $\mathcal{Y}$;*

(ii) $(t \cdot e) \circ t = (1_{\mathcal{Y}} \circ t)e$;

(iii) $t \cdot (e \circ t) = e(t \cdot 1_{\mathcal{Y}})$.

Construction based on double actions

Suppose that

(1) $\mathcal{X}$ and $\mathcal{X}'$ are semilattices;

(2) $\mathcal{Y}$ is a subsemilattice of both $\mathcal{X}$ and $\mathcal{X}'$;

(3) $\varepsilon \in \mathcal{X}$ and $\varepsilon' \in \mathcal{X}'$ such that $a \leq \varepsilon, \varepsilon'$ for all $a \in \mathcal{Y}$;

(4) T is a monoid with identity 1 and T acts via morphisms on the left of $\mathcal{X}$, via $\cdot$, and on the right of $\mathcal{X}'$, via $\circ$;

(5) for all $t \in T$, there exists $a \in \mathcal{Y}$ such that $a \leq t \cdot \varepsilon$.

Suppose that $\forall t \in T$ and $\forall e \in \mathcal{Y}$, the following hold:

(A) $e \leq t \cdot \varepsilon \Rightarrow e \circ t \in \mathcal{Y}$;

(B) $e \leq \varepsilon' \circ t \Rightarrow t \cdot e \in \mathcal{Y}$.

(C) $e \leq t \cdot \varepsilon \Rightarrow t \cdot (e \circ t) = e$;

(D) $e \leq \varepsilon' \circ t \Rightarrow (t \cdot e) \circ t = e$.

Construction based on double actions

Let us define

$$M = \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y}) = \{(a,t) \in \mathcal{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation

$$(a,t)(b,u) = (a \wedge t \cdot b, tu)$$

for $(a,t),(b,u) \in \mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$.

Construction based on double actions

Proposition

If T is an arbitrary monoid, then $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is proper restriction.

Proposition

If T is a unipotent monoid, then $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is proper weakly ample.

Proposition

If T is a cancellative monoid, then $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is proper ample.

Proposition

If T is a group, then $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$ is a proper inverse semigroup. Conversely, every proper inverse semigroup is isomorphic to some $\mathcal{M}(T, \mathcal{X}, \mathcal{X}', \mathcal{Y})$, where T is a group.

Construction based on partial actions

Suppose

(1) $\mathcal{Y}$ is a semilattice;

(2) T is a monoid, which acts partially on the right and left of $\mathcal{Y}$ (denoted by $\circ$ and $\cdot$ respectively);

(3) T preserves the partial orders;

(4) The domain of each $t \in T$ is an order ideal.

Suppose that for $e \in \mathcal{Y}$ and $a \in T$, the following hold:

(A) If $\exists e \circ a$, then $\exists a \cdot (e \circ a)$ and $a \cdot (e \circ a) = e$;

(B) If $\exists a \cdot e$, then $\exists (a \cdot e) \circ a$ and $(a \cdot e) \circ a = e$;

(C) For all $t \in T$, $\exists e \in \mathcal{Y}$ such that $\exists e \circ a$.

Construction based on partial actions

Let us define

$$M = \mathcal{M}(T, \mathcal{Y}) = \{(e, a) \in \mathcal{Y} \times T : \exists e \circ a\},$$

with binary operation

$$(e, a)(f, b) = (a \cdot (e \circ a \wedge f), ab)$$

for $(e, a), (f, b) \in \mathcal{M}(T, \mathcal{Y})$.

Structure Theorems

Theorem

If T is an arbitrary monoid, $M = \mathcal{M}(T, \mathcal{Y})$ is a proper restriction semigroup and $M/\sigma \cong T$. Conversely, every proper restriction semigroup S is isomorphic to some $\mathcal{M}(T, \mathcal{Y})$, where $S/\sigma \cong T$.

Theorem

$M = \mathcal{M}(T, \mathcal{Y})$ is a proper weakly ample semigroup if and only if T is unipotent.

Theorem (Lawson)

$M = \mathcal{M}(T, \mathcal{Y})$ is a proper ample semigroup if and only if T is right cancellative.

Theorem (Petrich, Reilly)

$M = \mathcal{M}(T, \mathcal{Y})$ is a proper inverse semigroup if and only if T is a group.

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