ON SOME MONOIDS ASSOCIATED TO COXETER GROUPS

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I don't need to convince anyone at this conference that spending one's time studying consequences of the Associative Law is a way to live a good and happy life. However, you may have found that your colleagues in neighbo(u)ring fields tend to know next to nothing about semigroups and in fact seem not to understand why an otherwise intelligent person like yourself would spend time studying such things.

Well, in all honesty, there is some justification in their perception. There are simply too many semigroups and a detailed study of all of them will invariably lead to work that is not connected much to the rest of mathematics. Here we look for semigroups that arise naturally in group theory and geometry.

In this talk, we will try to give you some ammunition to talk with your group theory friends about a number of monoids that are playing an important role in the geometry, algebra and combinatorics of Coxeter groups. We argue that the monoids we discuss are an essential tool in understanding the group itself and its concrete representation as a reflection group. They are necessary, because there are some important structures, such as Bruhat order, that can not be supported by a group itself and an associated monoid is necessary to give some algebraic tools to aid in the study of the group and its actions.

We will look at two examples. We look at the case of finite Coxeter groups for simplicity of exposition, but what we say is more general. As mentioned above, every Coxeter group has a partial order, called Bruhat order associated with it. Of course, no non-trivial finite group can be partially ordered, so there is no way to give a finite Coxeter group W a non-trivial multiplicative partial order.

Bruhat order is defined by subword ordering of normal forms in W. Simon's Theorem says in Algebraic Automata Theory says that subword ordering is intimately connected with finite J-trivial monoids and with monoids multiplicatively ordered in which the identity is the minimal element, so one might guess that there is some connection. We define a monoid $M(W)$ on the same underlying set of W which is in fact J-trivial and is multiplicatively ordered by the Bruhat order. This monoid has been discovered many times in a number of guises and we discuss its relationships with W. One beautiful definition is that $M(W)$ is obtained from the usual braid form of the Coxeter presentation of W by simply changing relations of the form $ss = 1$ for Coxeter generator s to $ss = s$.

In his epic work on buildings and groups with BN-pairs, Tits defined a certain operation called projection on the Coxeter complex $C(W)$ of W. Recall that $C(W)$ is the hyperplane arrangement one uses to define W concretely as a reflection group. It was not noticed for some time, that the Tits projections turn $C(W)$ into a left regular band (LRB), that is a band in which $J = L$ or equivalently a semigroup satisfying the identities $x^2 = x$ and $xyx = xy$.

The LRB structure on $C(W)$ carries a great deal of geometric and combinatorial information on W, which once again can not be seen directly from the group structure on W alone. In particular, the usual action of W on $C(W)$ is by automorphisms of the LRB structure and the invariant subalgebra of the monoid algebra of $C(W)$ is anti-isomorphic to the Descent Algebra of W, an algebra with very important applications to many problems. The original proof of Solomon that the Descent Algebra was in fact an algebra (it was originally defined by taking a certain linearly independent subset of the group algebra of W and showing that the product of any two was a linear combination of the elements in this collection) was quite difficult, whereas the anti-isomorphism is straightforward and clearly the invariant algebra is in fact an algebra.

This allows us to apply the representation theory of LRBs to understand the algebra of $C(W)$ and the Descent Algebra. In fact, like any band, the algebra of $C(W)$ is basic over any field K, that is, its semisimple part is a direct product of copies of K, one copy for each J-class of $C(W)$ and thus this is true for the Descent Algebra as well. We compute the Quiver of an LRB algebra showing that in fact LRBs are generalized Coxeter complexes.

All of this work is due to a number of other mathematicians whose names and papers will be mentioned in detail during the talk.

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