Pseudo-finite semigroups and diameter

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This is joint work with (variously) Yang Dandan, James East, Craig Miller, Thomas Quinn-Gregson, Nik Ruškuc and Rida e-Zenab.

Outline of talk

- Finitary conditions
- Right congruences
- Pseudo-finite monoids: background and motivation
- Minimal ideals in pseudo-finite monoids
- The positive results
- The negative results
- Diameter

Throughout, S will denote a monoid.

Finitary conditions
What are they? Why are these good things?

Let $\mathcal{A}$ be a class of algebras.

By algebra, I mean a **universal algebra**. The class $\mathcal{A}$ could be groups, rings, vector spaces over a given field, monoids,....

Definition

A **finitary** condition for $\mathcal{A}$ is a condition defined for algebras in $\mathcal{A}$ that is certainly satisfied for all finite $A \in \mathcal{A}$.

Of course, we hope our condition is satisfied for some **infinite** $A \in \mathcal{A}$.

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

Finitary condition (for a monoid)

A condition satisfied by all finite monoids.

Every element of S has an idempotent power.
For S we have $\mathcal{D} = \mathcal{J}$.

Right ideals of monoids do not correspond to (cyclic) right actions.

(Weakly) right noetherian monoid

A monoid S is weakly right noetherian if every right ideal is finitely generated.
A monoid S is **right noetherian** if every right congruence is finitely generated.

Right congruences

Right congruences

A **right congruence** on a monoid S is an equivalence relation ρ such that for every $a, b, c \in S$

$$a \,\rho\, b \Rightarrow ac \,\rho\, bc.$$

A relation ρ on S is a subset of $S \times S$; we pass without mention between $a \,\rho\, b$ and $(a, b) \in \rho$.

- Right congruences correspond to cyclic right actions.
- Right congruences form a sublattice of the lattice of equivalence relations on S.
- How do we generate right congruences?

Right congruences
Generation

If $U \subseteq S \times S$, then there is a smallest right congruence $\langle U \rangle$ containing U, the **right congruence generated by** U.

Explicit form of $\langle U \rangle$

We have $a \langle U \rangle b$ if and only if $a = b$ or there exists a **derivation**

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \cdots,\ d_n t_n = b,$$

where $(c_i, d_i) \in U \cup U^{-1}$ and $t_i \in S$.
A sequence as above is a U-**sequence of length** n**.**

Pseudo-finiteness and diameter

A monoid S is **pseudo-finite** if $\omega := S \times S$ is finitely generated as a right congruence and there is an upper bound on the length n of the derivations required.
If S is pseudo-finite, the smallest upper bound (over all finite generating sets) is the **diameter** of S.

The notion of pseudo-finiteness for S-acts and for semigroups is analogous.

Pseudo-finite monoids
Background and motivation

- Pseudo-finite monoids were introduced by Dales and White (2017) to understand the relation between maximal ideals in semigroup algebras being finitely generated, and the algebra itself being finitely generated.
- Kobayashi (2007) showed ω being finitely generated is equivalent to S being right FP_1.
- Right (actually, left) FP_1 monoids are investigated in Kobayashi (2007), Pride & Gray (2011) and G., Quinn-Gregson, Zenab & Yang (2019). In the latter paper we also considered the case of pseudo-finiteness.

- The right diagonal act $S \times S$ of S has action

$$(u, v)t = (ut, vt).$$

  If S has finitely generated right diagonal act $S \times S$, then S is pseudo-finite.
- If S is right noetherian then certainly S is right FP_1.
- Pseudo-finiteness is a finitary condition, in that clearly all finite monoids are pseudo-finite (take $U = S \times S$).
- The use of finitary conditions have become embedded in standard algebraic practice over the last century. Where does pseudo-finiteness fit?

Minimal ideals in pseudo-finite monoids
A few observations

A finite monoid S is pseudo-finite: take $U = S \times S$.

A finite monoid has a minimal ideal!!

Any monoid S with zero is pseudo-finite.

Take $U = \{(1, 0)\}$. Then for any $a, b \in S$ we have

$$a = 1a,\ 0a = 0b,\ 1b = b.$$

Certainly a monoid with zero has a minimal ideal.

Dales and White conjectured that any pseudo-finite monoid is exactly $M \times F$ where M has a zero and F is finite.

Their conjecture was incorrect (G., Quinn-Gregson, Yang and Zenab (2019)). Nevertheless they had realised the existence and behaviour of minimal ideals in pseudo-finite monoids is important.

Fact

A group G is such that ω is finitely generated if and only if G is a finitely generated group.

Corollary (to proof)

A group G is pseudo-finite if and only if it is finite.

Fact: White (2017)

A left cancellative monoid is pseudo-finite if and only if it is a **finite** group.

Any group has a minimal ideal.

Minimal ideals in pseudo-finite monoids
General observation

Theorem: GMQ-GR

A monoid S is pseudo-finite if and only if it has a right ideal that is pseudo-finite as a right S-act.

If I is my right ideal and $u \Rightarrow v$ for all $u, v \in I$, via U, then add $(1, k)$ to U where $k \in I$. Then for any $a, b \in S$ we have

$$a = 1a,\ ka \Rightarrow\ kb, 1b = b.$$

Consequently, any monoid with a finite minimal ideal is pseudo-finite.

Minimal ideals in pseudo-finite monoids

Any minimal ideal of a semigroup is simple (i.e. has no proper ideals).

G., Quinn-Gregson, Yang and Zenab (2019) gave necessary and sufficient conditions for a monoid with a minimal completely simple ideal to have ω finitely generated.

Gray-Pride (2006) had proven this in a special case.

We posed the question of whether every pseudo-finite monoid must have a minimal completely simple ideal.

Theorem: Miller 2020

The Baer-Levi semigroup BL(p, p) is pseudo-finite (but not completely simple).

Questions?

Which classes of pseudo-finite monoids have minimal ideals, and of what kind?

Theorem: GMQ-GR

A monoid with a minimal completely simple ideal $\mathcal{M} = \mathcal{M}(G; I, J; P)$ is pseudo-finite if and only if it satisfies (A) and (B).

(A) A condition on generating G;

(B) S makes J into a pseudo-finite act.

(A) there exists a left ideal K_0 of K such that K_0 is the union of finitely many $\mathcal{L}$-classes and any maximal subgroup $G = H_e$ of K_0 has finite $(F \cup V)$-diameter, where $F \subseteq G$ is finite and

$$V = \{fg : f, g \in E(K_0), f \,\mathcal{R}\, e \,\mathcal{L}\, g\} \subseteq G.$$

Consequently

- for any group H, there is an $\mathcal{M}(H; I, J; P)$ such that $\mathcal{M}(H; I, J; P)^1$ is pseudo-finite;
- there is a $\mathcal{M}(G; I, J; P)$ that satisfies (B) but not (A)
  and
- there is a $\mathcal{M}(G; I, J; P)$ that satisfies (A) but not (B).

The positive results: when pseudo-finite monoids have minimal ideals - completely simple case

Theorem: G., Quinn-Gregson, Yang and Zenab (2019)

Let S be an inverse monoid. Then S is pseudo-finite if and only if S has a minimal ideal G where G is a finite group.

Theorem: GMQ-GR

Let S be a commutative monoid. Then S is pseudo-finite if and only if S has a minimal ideal G where G is a finite group.

Theorems: GMQ-GR

We extended this to right reversible monoids, where S is **right reversible** if for any $a, b \in S$ we have $c, d \in S$ such that $ca = db$; to completely regular monoids (monoids that are unions of groups) and to orthodox monoids (regular monoids with band of idempotents).

The positive results when pseudo-finite monoids have minimal ideals - Any minimal ideal

Theorem: GMQ-GR

Let S be pseudo-finite. Then S has a minimal ideal if and only if S contains an ideal I such that $\leq_{\mathcal{J}} \cap (I \times I)$ is left compatible with multiplication in S.

Corollary: GMQ-GR

Let S be pseudo-finite such that $\leq_{\mathcal{J}}$ is left compatible. Then S has a minimal ideal.

The negative results

Recall that Miller showed there exists a pseudo-finite monoid with minimal ideal that is not completely simple.

Example: GMQ-GR

Gave an example of an infinite semigroup of mappings that is pseudo-finite but does not have a minimal ideal.

The negative results

Example: GMQ-GR

Gave an example of an regular semigroup that is pseudo-finite but does not have a minimal ideal.

This was done via a certain construction:

Construction: GMQ-GR

From ingredients U, T (semigroups) I, J (sets) satisfying certain conditions one obtains an ideal extension S of $\mathcal{M}(T; I, J; P)$ by S^1 that is guaranteed to be pseudo-finite.

Theorem: GMQ-GR

There exists a $\mathcal{J}$-trivial pseudo-finite monoid with no minimal ideal.

Diameter
Diagonal acts - again

Diagonal acts

Let S be a semigroup. Then $S \times S$ with action

$$(u, v)t = (ut, vt)$$

is the **(right) diagonal act** of S.

If $S \times S$ is generated by U, then for any $a, b \in S$ we have $(a, b) = (u, v)t$ for some $(u, v) \in U$ and then

$$a = ut \to vt = b, \text{ some } t \in S.$$

Corollary

A semigroup S has diameter 1 if and only if its diagonal act is finitely generated.

Diameter

Corollary: Robertson, Ruškuc & Thompson; Gallagher

For any infinite set X the monoids $\mathcal{T}_X, \mathcal{B}_X$ and $\mathcal{PT}_X$ each has diameter 1.

Theorem: Gallagher and Ruškuc

For any infinite set X the monoid $\mathcal{I}_X$ does not have diameter 1.

So: as $\mathcal{I}_X$ has a zero,

$\mathcal{I}_X$ has diameter 2.

We - GMQ-GR and James East - are conducting an investigation of diameters of ‘natural’ semigroups of transformations.

Let p be an infinite cardinal and let $|X| = p$. The monoid Inj_X consists of all injective maps $X \to X$.
The Baer-Levi semigroup $BL(p, p)$ consists of all $\alpha \in \mathrm{Inj}_X$ such that $|X \setminus X\alpha| = p$. The semigroup $BL(p, p)$ is right cancellative, right simple and is the minimal ideal of Inj_X.

Theorem: the team

The diameter of $BL(p, p)$ is 3.

Since $BL(p,p)$ is the minimal ideal of Inj_X where $X = p$, we have:

Corollary

Let X be infinite. The diameter of Inj_X is 3,4 or 5.

Theorem: the team

Let X be infinite. The diameter of Inj_X is 4.

Where to now?

- What are the diameters of other natural transformation semigroups? What about the left diameters?
- Diameters in the context of minimal ideals and other algebraic properties.

Theorem

There is a pseudo-finite regular semigroup with diameter 3 and no minimal ideal.

- Which simple semigroups are minimal ideals of pseudo-finite monoids?
- Homological condition for pseudo-finiteness.
- Can we describe semigroups for which every right congruence of finite index is finitely generated in a bounded way?
- This work is taking place in the general context of finite generation of right congruences...

Thanks!! And references -

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