The $\mathcal{R}$-height of Semigroups and their Bi-ideals

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Outline

1. Definitions and basic facts

2. Main Results
   - Bounds
   - Can the bounds be attained?

$\mathcal{R}$-height

Green's preorder $\leq_{\mathcal{R}}$ is defined by

$$a \leq_{\mathcal{R}} b \Leftrightarrow aS^1 \subseteq bS^1.$$

We write $a \leq_S b$ for $a \leq_{\mathcal{R}} b$, and $a <_S b$ if $a \leq_{\mathcal{R}} b$ but $aS^1 \neq bS^1$.

The pre-order $\leq_{\mathcal{R}}$ induces a partial order on the set of $\mathcal{R}$-classes of S, given by $R_a \leq R_b \Leftrightarrow a \leq_S b$.

The $\mathcal{R}$**-height** of S, denoted by $H_{\mathcal{R}}(S)$, is the height of the poset $S/\mathcal{R}$, i.e. the supremum of the lengths of chains of $\mathcal{R}$-classes of S.

Bi-ideals

A **bi-ideal** of S is a subsemigroup B such that $BSB \subseteq B$.

Bi-ideals include right ideals and left ideals (and hence ideals).

- If B is a bi-ideal and T is a subsemigroup of S, and $C = B \cap T \neq \emptyset$, then C is a bi-ideal of T.
- The intersection of bi-ideals is either empty or a bi-ideal.
- If B is a bi-ideal and X is any subset of S, then BX and XB are bi-ideals of S.
- Bi-ideals of right simple semigroups are left ideals.

Minimal ideals

A **minimal** (**right**) **ideal** is a (right) ideal that contains no proper (right) ideal.

If it exists, the minimal ideal of S, also known as the **kernel** of S, will be denoted by $K(S)$.

If S has min. right ideals, then $K(S)$ is the union of all the min. right ideals. If S additionally has min. left ideals, then $K(S)$ is completely simple.

Lemma. If $H_{\mathcal{R}}(S)$ is finite, then S has minimal right ideals. Moreover, $H_{\mathcal{R}}(S) = 1$ if and only if S is a union of minimal right ideals.

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Bi-ideals

Let S be a semigroup with finite $\mathcal{R}$-height, and let B be a bi-ideal of S. Let n denote the maximum length of a chain of $\mathcal{R}$-classes of S that intersect B.

Theorem. $H_{\mathcal{R}}(B) \leq 3n - 1$.

Theorem. If $K(S)$ is completely simple, then $H_{\mathcal{R}}(B) \leq 3n - 2$.

Theorem. If every element of B has a local right identity (i.e. $bB \subseteq B$ for all $b \in B$), then $H_{\mathcal{R}}(B) = n$.

Left ideals

Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a left ideal of S. Let n denote the maximum length of a chain of $\mathcal{R}$-classes of S that intersect A.

Theorem. $H_{\mathcal{R}}(A) \leq 2n$.

Theorem. If $K(S)$ is completely simple, then $H_{\mathcal{R}}(A) \leq 2n - 1$.

Theorem. If $A \subseteq \text{Reg}(S)$, then $H_{\mathcal{R}}(A) = n$.

Right ideals and two-sided ideals

Let S be a semigroup with finite $\mathcal{R}$-height, and let A be a right ideal of S. Let n denote the maximum length of a chain of $\mathcal{R}$-classes of S *contained in* A.

Theorem. $H_{\mathcal{R}}(A) \leq 2n - 1$.

Theorem. If A is a two-sided ideal, then $H_{\mathcal{R}}(A) \leq n$.

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Problems

- For each $n \in \mathbb{N}$, does there exist a semigroup S and a bi-ideal B of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 1$?
- For each $n \in \mathbb{N}$, does there exist a semigroup S with a completely simple kernel and a bi-ideal B of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 2$?
- For each $n \in \mathbb{N}$, does there exist a semigroup S and a left ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n$?
- For each $n \in \mathbb{N}$, does there exist a semigroup S with a completely simple kernel and a left ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?
- For each $n \in \mathbb{N}$, does there exist a semigroup S and a right ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?
- For each $n \in \mathbb{N}$, does there exist a semigroup S with an ideal A such that $H_{\mathcal{R}}(S) = H_{\mathcal{R}}(A) = n$? ✓

Bi-ideal: 3n-2 bound

Theorem. Let $n \geq 2$. Let S be defined by the presentation

$$\langle x, y, z, t \mid xyzt = x,\ yzty = y,\ ztyz = z,\ tyzt = t,\ w = 0 \\ (w \in \{x^n, y^2, z^2, t^2, xz, xt, yx, yt, zx, zy, tz, tx^{n-1}\})\rangle$$

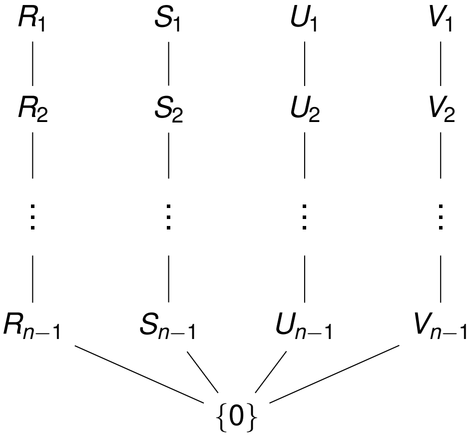
Let $B = X \cup XS^1X$ where $X = \{x, y, z, tx\}$. Then $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(B) = 3n - 2$.

$$S = \left(\bigcup_{i=1}^{n-1} (R_i \cup S_i \cup U_i \cup V_i)\right) \cup \{0\},$$

where $R_i = \{x^i, x^iy, x^iyz\}$, $S_1 = \{y, yz, yzt\}$, $S_j = yztR_{j-1}, U_1 = \{z, zt, zty\}$, $U_j = ztR_{j-1}$, $V_1 = \{t, ty, tyz\}$, $V_j = tR_{j-1}$ $(2 \leq j \leq n-1)$.

$B = S \setminus \{yzt, zt, t, ty, tyz\}$.

Poset of $\mathcal{R}$-classes of S



$n = 2$ case

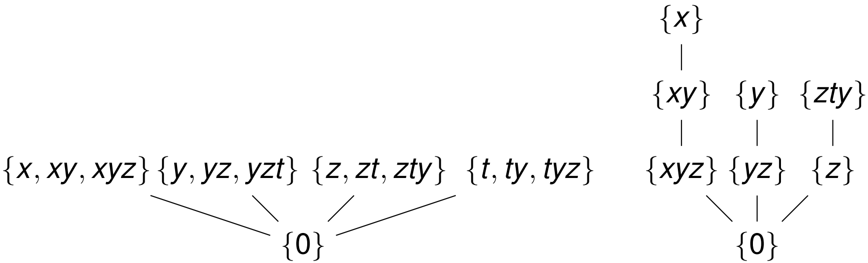


Figure: The poset of $\mathcal{R}_S$-classes (left) and the poset of $\mathcal{R}_B$-classes (right)

Left ideal: 2n bound

For each $n \in \mathbb{N}$, does there exist a semigroup S and a left ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n$?

Proposition. Let S be a right simple semigroup (so $H_{\mathcal{R}}(S) = 1$) that is not completely simple, and let A be a principal left ideal $S^1 a$. Then the $\mathcal{R}$-classes of A are $\{a\}$ and $A \setminus \{a\} = Sa$, and hence $H_{\mathcal{R}}(A) = 2$.

Theorem. Let $n \geq 2$. Let S be a semigroup with a left ideal A such that $H_{\mathcal{R}}(S) = n - 1$ and $H_{\mathcal{R}}(A) = 2(n - 1)$. Let T be any right simple semigroup that is not completely simple, and let U be the semigroup defined by the presentation

$$\langle S, T \mid ab = a \cdot b,\ cd = c \cdot d,\ ac = c\ \ (a, b \in S, c, d \in T) \rangle.$$

Fix $c \in T$, and let $B = T^1(A \cup \{c\})$. Then $H_{\mathcal{R}}(U) = n$ and $H_{\mathcal{R}}(B) = 2n$.

$U = S \cup T \cup TS$ and $K(U) = T \cup TS$.

Posets of $\mathcal{R}$-classes

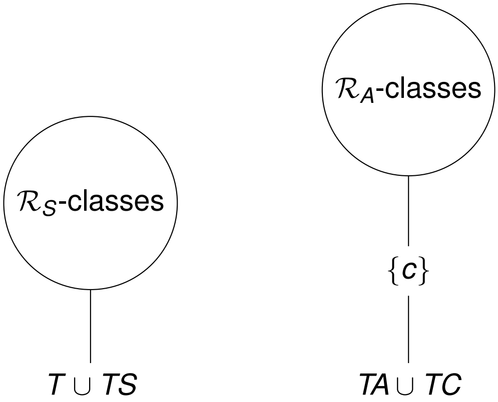


Figure: The poset of $\mathcal{R}_U$-classes (left) and the poset of the $\mathcal{R}_B$-classes (right).

Left ideal: 2n-1 bound

For each $n \in \mathbb{N}$, does there exist a semigroup S with a completely simple kernel and a left ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?

Theorem. Let $n \geq 2$. Let S be defined by the presentation

$$\langle x, y, z \mid xyz = x,\ yzy = y,\ zyz = z,\ u = 0\,(u \in \{x^n, y^2, z^2, xz, yx, zx^{n-1}\}\rangle$$

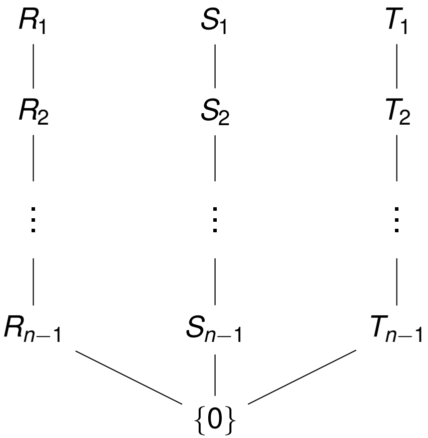
and let $A = S^1\{x, y\}$. Then $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$.

$$S = \left(\bigcup_{i=1}^{n-1} (R_i \cup S_i \cup T_i)\right) \cup \{0\},$$

$$R_i = \{x^i, x^i y\},\ \ S_1 = \{y, yz\},\ \ S_j = yzR_{j-1}, T_1 = \{z, zy\},\ \ T_j = zR_{j-1}.$$

$$A = S \backslash \{yz, z\}.$$

Poset of $\mathcal{R}$-classes of S



$n = 2$ case

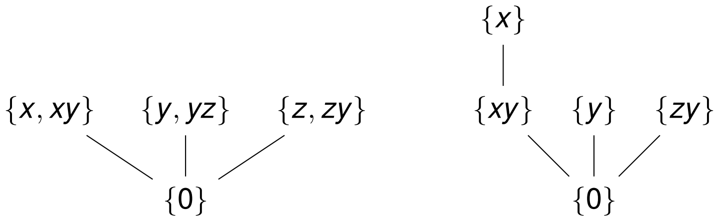


Figure: The poset of $\mathcal{R}_S$-classes (left), and the poset of $\mathcal{R}_A$-classes (right).

Right ideal: 2n-1 bound

For each $n \in \mathbb{N}$, does there exist a semigroup S and a right ideal A of S such that $H_{\mathcal{R}}(S) = n$ and $H_{\mathcal{R}}(A) = 2n - 1$?

Let S be a semigroup and let I be a non-empty set. The *Brandt extension of S by I*, denoted by $\mathcal{B}(S, I)$, is the semigroup with universe $(I \times S \times I) \cup \{0\}$ and multiplication given by $0x = x0 = 0$ and

$$(i, s, j)(k, t, l) = \begin{cases} (i, st, l) & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Theorem. Let $n \geq 2$. Let S be a semigroup with a right ideal A of S such that $H_{\mathcal{R}}(S) = n - 1$ and $H_{\mathcal{R}}(A) = 2(n-1) - 1$. Let I be any set with $|I| \geq 2$, and let $T = \mathcal{B}(S, I)$. Fix $1 \in I$ and let

$$B = (1, a, 1)T^1 = (\{1\} \times A \times I) \cup \{0\}.$$

Then $H_{\mathcal{R}}(T) = n$ and $H_{\mathcal{R}}(B) = 2n - 1$.

5-element Brandt semigroup

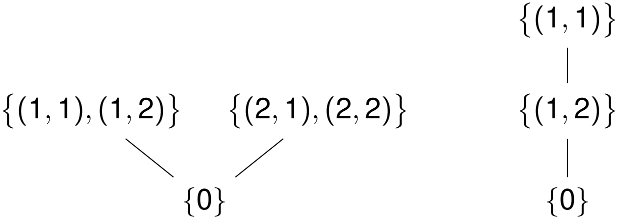


Figure: The poset of $\mathcal{R}$-classes of the 5-element Brandt semigroup S (left), and the poset of the $\mathcal{R}$-classes of the principal right ideal $A = (1,1)S^1$ (right).

Thanks for listening