Tensor products and preservation of weighted limits, for S-posets

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Def 1 Let S be a partially ordered monoid (shortly pomonoid). A **right** S-**poset** is a poset A together with an action $A \times S \to A$, $(a, s) \mapsto as$, such that

1. $(as)t = a(st)$,
2. $a1 = a$,
3. $a \leq a' \Longrightarrow as \leq a's$,
4. $s \leq t \Longrightarrow as \leq at$

for every $a, a' \in A$, $s, t \in S$.

Similarly left S-posets are defined. S-poset morphisms are order and action preserving mappings. Right (left) S-posets and their morphisms form a category Pos_S ($_S\mathsf{Pos}$), where isomorphisms are surjective mappings that preserve and reflect order.

The category $_S\mathsf{Pos}$ (similarly Pos_S) is a Pos-category (or a category enriched over the category Pos of posets), where the morphism sets $_S\mathsf{Pos}(A, B)$, $_SA, {_SB} \in {_S\mathsf{Pos}}$ are posets with respect to pointwise order.

If $\mathcal{A}$ and $\mathcal{B}$ are Pos-categories then a Pos-functor $F : \mathcal{A} \to \mathcal{B}$ has to preserve the order of morphism posets. We shall call such functors **pofunctors**. If $\mathcal{A}$ and $\mathcal{B}$ are Pos-categories, $\mathcal{A}$ is small and $F, G : \mathcal{A} \to \mathcal{B}$ are pofunctors then the set $\mathsf{Nat}(F, G)$ of natural transformations from F to G is a poset with respect to the order

$$(\alpha_A)_{A\in\mathcal{A}} \leq (\beta_A)_{A\in\mathcal{A}} \Longleftrightarrow \alpha_A \leq \beta_A \text{ for every } A \in \mathcal{A} \text{ in the poset } \mathcal{B}(F(A), G(A)).$$

In enriched categories (or 2-categories) one can consider weighted (or indexed) limits. In the case of the category ${}_S\mathsf{Pos}$, this general definition takes the following form.

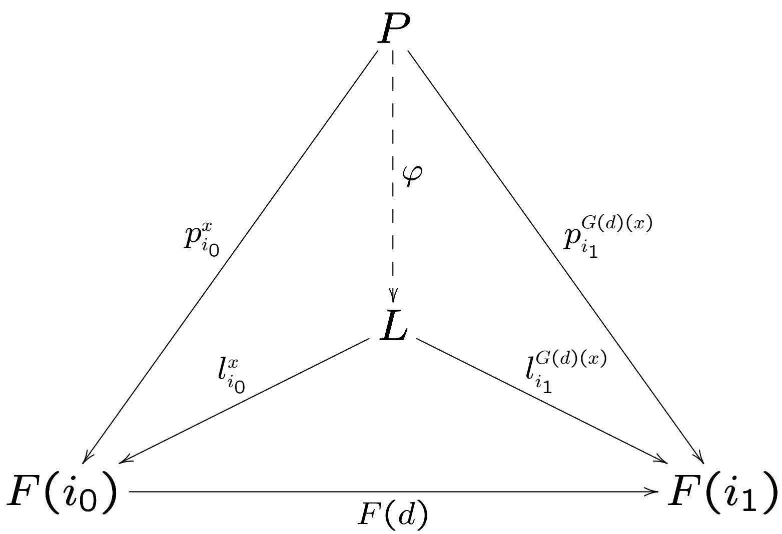
Def 2 Let S be a pomonoid, $\mathcal{D}$ a small Pos-category with the object set I, $F : \mathcal{D} \to {}_S\mathsf{Pos}$ and $G : \mathcal{D} \to \mathsf{Pos}$ pofunctors. A Pos-**limit of** F **weighted by** G is a pair $\left({}_SL, (l_i^x)_{i\in I}^{x\in G(i)}\right)$, where $l_i^x : L \to F(i)$ are left S-poset morphisms and

1. (a) $x \leq x'$ implies $l_i^x \leq l_i^{x'}$ for every $i \in I$ and $x, x' \in G(i)$;

   (b) $F(d)l_{i_0}^x = l_{i_1}^{G(d)(x)}$ for every $d : i_0 \to i_1$ in $\mathcal{D}$ and $x \in G(i_0)$;

2. for every ${}_SP \in {}_S\mathsf{Pos}$ and family $(p_i^x)_{i\in I}^{x\in G(i)}$ of left S-poset morphisms $p_i^x : P \to F(i)$ with properties 1, there is a unique left S-poset morphism $\varphi : P \to L$ such that $l_i^x\varphi = p_i^x$ for every $i \in I$ and $x \in G(i)$.

We write $\left({}_S L, (l_i^x)_{i\in I}^{x\in G(i)}\right) \approx \lim_G F$.



Weighted limits always exist in the category ${}_S\mathsf{Pos}$ (or Pos, which is just ${}_{\{1\}}\mathsf{Pos}$), as shown by the following canonical construction.

It is easy to see that the poset $\mathsf{Nat}(G, UF)$, where $U : {}_S\mathsf{Pos} \to \mathsf{Pos}$ is the forgetful functor, is an S-poset if the left S-action is given by

$$sf := (sf_i)_{i\in I},$$

where $s \in S$, $f = (f_i)_{i\in I} \in \mathsf{Nat}(G, UF)$, and the mapping $sf_i : G(i) \to F(i)$ is defined by

$$(sf_i)(x) := sf_i(x),$$

$x \in G(i)$. For every $i \in I$ and $x \in G(i)$ we define a mapping $l_i^x : \mathsf{Nat}(G, UF) \to F(i)$ by

$$l_i^x(f) := f_i(x),$$

$f = (f_i)_{i\in I} \in \mathsf{Nat}(G, UF)$.

Proposition 1 *The pair* $\left(\mathsf{Nat}(G, UF), (l_i^x)_{i\in I}^{x\in G(i)}\right)$ *is a* Pos*-limit of F weighted by G.*

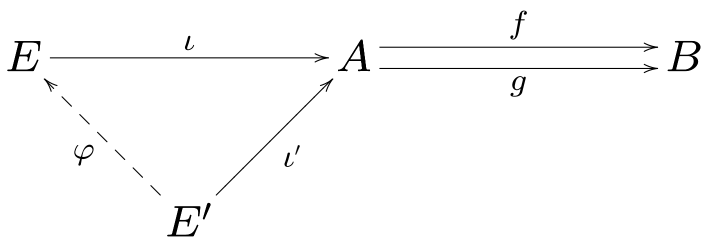
All usual limits (e.g. products, equalizers) are instances of weighted limits. We shall also need inserters and comma-objects, which have been introduced in [9] for arbitrary 2-categories. Note that inserters (comma-objects) in ${}_S\mathsf{Pos}$ were called subequalizers (subpullbacks) in [4].

Def 3 An **inserter** of a pair (f, g) of morphisms $A \to B$ in ${}_S\mathsf{Pos}$ is a pair (E, ι), where $\iota \in {}_S\mathsf{Pos}(E, A)$ is such that

1. $f\iota \leq g\iota$,
2. if $\iota' \in {}_S\mathsf{Pos}(E', A)$ is another morphism such that $f\iota' \leq g\iota'$ then there exists a unique morphism $\varphi \in {}_S\mathsf{Pos}(E', E)$ such that $\iota\varphi = \iota'$.

As a canonical inserter one can take

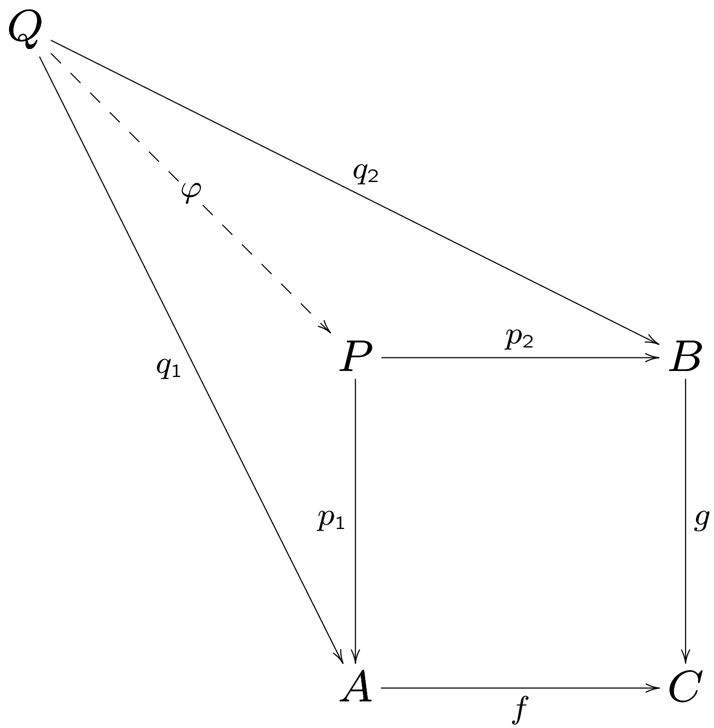
$$E := \{a \in A \mid f(a) \leq g(a)\} \subseteq A.$$



Def 4 A **comma-object** of a pair (f, g) of morphisms $f \in {}_S\mathsf{Pos}(A, C)$, $g \in {}_S\mathsf{Pos}(B, C)$ is a triple (P, p_1, p_2), where $p_1 \in {}_S\mathsf{Pos}(P, A)$, $p_2 \in {}_S\mathsf{Pos}(P, B)$ are such that

1. $fp_1 \leq gp_2$,

2. if $q_1 \in {}_S\mathsf{Pos}(Q, A)$, $q_2 \in {}_S\mathsf{Pos}(Q, B)$ are another morphisms such that $fq_1 \leq gq_2$ then there exists a unique morphism $\varphi \in {}_S\mathsf{Pos}(Q, P)$ such that $p_1\varphi = q_1$ and $p_2\varphi = q_2$.



If $A_S \in \mathsf{Pos}_S$ and ${}_SB \in {}_S\mathsf{Pos}$ then we can consider a preorder θ on the set $A \times B$, defined by $(a,b)\theta(a',b')$ if and only if $(a,b) = (a',b')$ or

$$\begin{array}{rclrcl} a & \leq & a_1 s_1 & & & \\ a_1 t_1 & \leq & a_2 s_2 & s_1 b & \leq & t_1 b_2 \\ & \cdots & & & \cdots & \\ a_n t_n & \leq & a' & s_n b_n & \leq & t_n b' \end{array}$$

for some $a_i \in A$, $b_i \in B$, $s_i, t_i \in S$. Then $\theta \cap \theta^{-1}$ is an equivalence relation and

$$A \otimes_S B := (A \times B)/(\theta \cap \theta^{-1}) = \{a \otimes b \mid a \in A, b \in B\}$$

is a poset with order

$$a \otimes b \leq a' \otimes b' \Longleftrightarrow (a,b)\theta(a',b').$$

This poset $A \otimes_S B$ is called the **tensor product** of A_S and ${}_SB$. Note that

$$as \otimes b = a \otimes sb$$

for every $a \in A$, $b \in B$ and $s \in S$.

For a fixed S-poset A_S one can consider the pofunctor $A \otimes - : {}_S\mathsf{Pos} \to \mathsf{Pos}$ of tensor multiplication, defined by

$$\begin{array}{rcl} (A \otimes -)({}_SB) & := & A \otimes_S B, \\ (A \otimes -)(f) & := & 1_A \otimes f : A \otimes_S B \to A \otimes_S C : a \otimes b \mapsto a \otimes f(b), \end{array}$$

$f \in {}_S\mathsf{Pos}(B,C)$.

Def 5 We say that a right S-poset A_S is **limit flat** (**inserter flat**, **comma-object flat**, **product flat**) if the functor $A \otimes - : {}_S\mathsf{Pos} \to \mathsf{Pos}$ preserves small weighted limits (resp. inserters, comma-objects, small products).

Theorem 1 *The following assertions are equivalent for a non-empty right S-poset A_S:*

1. *A_S is limit flat;*
2. *A_S is inserter flat and product flat;*
3. *A_S is cyclic and satisfies the following condition: for every non-empty set K and all families $(s_k)_{k\in K}, (t_k)_{k\in K} \in S^K$*

   $$(E_\infty) \qquad (\forall k \in K)(as_k \leq at_k) \Rightarrow (\exists e \in S)(a = ae \wedge (\forall k \in K)(es_k \leq et_k));$$

4. *A_S is a cyclic projective.*

Next we consider preservation of certain finite weighted limits.

We shall use the following conditions on a right S-poset A_S that first appear in [4]:

(E) $(\forall a \in A)(\forall s, s' \in S)\,(as \leq as' \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \wedge us \leq us'))\,,$

(P) $(\forall a, a' \in A)(\forall s, s' \in S)(as \leq a's' \Rightarrow$
$(\exists a'' \in A)(\exists u, u' \in S)(a = a''u \wedge a' = a''u' \wedge us \leq u's')).$

An S-poset A_S is called **locally cyclic** if for every $a, a' \in A$ there exists $b \in A$ such that $a, a' \in bS$.

The notion of finite weighted (or indexed) limit is introduced in [7]. In the case of Pos-limits it sounds as follows.

Def 6 A weight $G : \mathcal{D} \to$ Pos is called **finite** if

1. $\mathcal{D}$ is a finite category,

2. $G(i)$ is a finite poset for every $i \in I$.

A **finite weighted limit** is one whose weight is finite.

For a functor $G : \mathcal{D} \to$ Pos we can consider its category of elements (or Grothendieck category). The objects of this category el(G) are pairs (x, i), where $i \in I$ and $x \in G(i)$. A morphism $(x, i) \to (y, j)$ is a morphism $d \in \mathcal{D}(i, j)$ such that $G(d)(x) = y$.

Among weighted limits, pie-weighted limits play an important role (see [11]).

Def 7 (11) A pofunctor $G : \mathcal{D} \to$ Pos is called a **pie weight** if each connected component of the category el(G) has an initial object.

Since equifiers (see [9] for the definition) are trivial in ${}_S$Pos and Pos, from Theorem 2.8 of [11] we have the following corollary.

Theorem 2 *A pofunctor $H : {}_S\mathsf{Pos} \to \mathsf{Pos}$ preserves finite pie-weighted limits if and only if it preserves finite products and inserters.*

We say that an S-poset A_S is **finite pie-limit flat** if the functor $A \otimes - : {}_S\mathsf{Pos} \to$ Pos preserves finite pie-weighted limits.

Def 8 Let $\varphi : B_S \to A_S$ be a surjective S-poset morphism. We say that φ is a 1-**pure epimorphism**, if

$$\begin{aligned} as_1 &\leq at_1, \\ &\dots \\ as_n &\leq at_n, \end{aligned} \tag{1}$$

$a \in A$, $s_1, \dots, s_n, t_1, \dots, t_n \in S$, implies that there exists $b \in B$ such that $\varphi(b) = a$ and

$$\begin{aligned} bs_1 &\leq bt_1, \\ &\dots \\ bs_n &\leq bt_n. \end{aligned}$$

Def 9 A nonempty category $\mathcal{D}$ is called **filtered**, if

1. for any objects i and i' there exist an object k and morphisms $d : i \to k$, $d' : i' \to k$;

2. for any morphisms $i \overset{d}{\underset{d'}{\rightrightarrows}} j$ there exists an object k and a morphism $f : j \to k$ such that $fd = fd'$.

Lemma 1 (Cf. [5], Theorem 1.2) *If θ is a preorder on an S-poset A_S compatible with action and extending the order of A (i.e. $a \leq a'$ implies $a\theta a'$) then $\sigma := \theta \cap \theta^{-1}$ is an S-poset congruence on A and A/σ is a right S-poset with respect to natural action and order given by*

$$[a]_\sigma \leq [a']_\sigma \Longleftrightarrow a\theta a'.$$

Proposition 2 *Let $\mathcal{D}$ be a small filtered category with the object set I and let $F : \mathcal{D} \to \mathsf{Pos}_S$ be a functor.*

1. *The relation θ, defined by*

$$a\theta a' \Longleftrightarrow (\exists j \in I)(\exists d : i \to j)(\exists d' : i' \to j)(F(d)(a) \leq F(d')(a')),$$

   $a \in F(i)$, $a' \in F(i')$, is a compatible order extending preorder on $\bigsqcup_{i \in I} F(i)$.

2. *If $\sigma = \theta \cap \theta^{-1}$ then, for every $a \in F(i)$ and $a' \in F(i')$,*

$$a\sigma a' \Longleftrightarrow (\exists j \in I)(\exists d : i \to j)(\exists d' : i' \to j)(F(d)(a) = F(d')(a')).$$

3. *A colimit of F can be constructed as a pair $(A, (\varphi_i)_{i \in I})$, where $A = \left(\bigsqcup_{i \in I} F(i)\right)/\sigma$ and the morphisms $\varphi_i : F(i) \to A$ are defined by $\varphi_i(x) := [x]$.*

For a subset $H \subseteq A \times A$ we introduce a binary relation $\beta(H)$ on A by setting $x\beta(H)y$ if and only if $x = y$ or there exist $h_1, \ldots, h_n, h'_1, \ldots, h'_n \in A$ and $s_1, \ldots, s_n$ such that

$$\begin{array}{ccccccccccccc} x & \leq & h_1 s_1 & & h'_2 s_2 & \leq & h_3 s_3 & & h'_{n-1} s_{n-1} & \leq & h_n s_n & & \\ & & h'_1 s_1 & \leq & h_2 s_2 & & & \cdots & & & h'_n s_n & \leq & y \end{array}$$

and $(h_i, h'_i) \in H$ for every $i = 1, \ldots, n$. Then the relation $\nu(H)$, defined by

$$x\nu(H)y \Longleftrightarrow x\beta(H)y \text{ and } y\beta(H)x$$

will be an S-poset congruence on A_S, which we call **the congruence induced by the set** H (see [4]). We write $\nu(a, a')$ for $\nu(\{(a, a')\})$.

Theorem 3 *The following assertions are equivalent for a non-empty right S-poset A_S:*

1. A_S *is finite pie-limit flat;*
2. A_S *is inserter flat and locally cyclic;*
3. A_S *is inserter flat, comma-object flat and locally cyclic;*
4. A_S *is locally cyclic and satisfies condition* (E)*;*
5. A_S *is locally cyclic and every surjective* S*-poset morphism* $B_S \to A_S$ *is a* 1*-pure epimorphism;*
6. A_S *is locally cyclic and every* S*-poset morphism* $S/\nu(H) \to A_S$*, where* H *is finite, factors through* S_S*;*
7. A_S *is a filtered colimit of* S*-posets that are isomorphic to* S_S*.*

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