Free Adequate Semigroups

Mark Kambites

School of Mathematics, University of Manchester

NBSAN St Andrews, 17 April 2009

Semigroup Theory

Philosophy

- *Semigroups are* **partly algebraic** *and* **partly combinatorial**.
- *Break them up into an algebraic bit (somebody else's problem)*
- *. . . and a combinatorial bit (somebody else's problem)*

Example (Krohn-Rhodes Theory)

- Algebraic part: groups
- Combinatorial part: aperiodic (group-free) semigroups
- Interplay: wreath products

Example ("Rees Theory")

- Algebraic part: groups
- Combinatorial part: eggboxes
- Interplay: the Rees matrix construction

Inverse Semigroups

Definition

A semigroup S is **inverse** if the idempotents commute and for every $x \in S$ there is an element y with $xyx = x$;

Idea

- *The existence of inverses forces a strong relationship between general elements and idempotents.*
- *If idempotents commute then their structure is (i) independent of the rest of the semigroup and (ii) essentially combinatorial rather than algebraic.*

Philosophy

- *local structure is group-like (somebody else's problem);*
- *global structure is semilattice-like (somebody else's problem);*
- *interplay is (sometimes) manageable.*

Rule

To understand a semigroup, we should seek:

- *a local invertible structure;*
- *a global combinatorial structure;*
- *a sufficient understanding of the relationship between them.*

Exception

Cancellative monoids *don't really decompose like this, but they are still relatively easy to understand.*

Idea (Fountain)

Replace "locally invertible" with "locally cancellative-like".

Question

What on earth does that mean?

Adequate Semigroups

Definition

A semigroup S is **left adequate** if idempotents commute and for each $a \in S$ there is an idempotent $e \in S$ such that $xa = ya \iff xe = ye$.

Definition

A semigroup S is **right adequate** if idempotents commute and for each $a \in S$ there is an idempotent $e \in S$ such that $ax = ay \iff ex = ey$.

Definition

A semigroup is **adequate** if it is both left and right adequate.

Philosophy

- *local structure is "cancellative-like";*
- *global structure is semilattice-like;*
- *interplay is (occasionally) manageable.*

The $+$ and $*$ Operations

Proposition

Let S be a left adequate semigroup. For each $a \in S$ there is a **unique** *idempotent a^+ such that $xa = ya$ if and only if $xa^+ = ya^+$.*

Proposition

Let S be a right adequate semigroup. For each $a \in S$ there is a **unique** *idempotent a^* such that $ax = ay$ if and only if $a^*x = a^*y$.*

Remark

The operations $x \mapsto x^+$ and $x \mapsto x^*$ are so fundamental that we consider left/right/two-sided adequate semigroups as algebras of signature $(2, 1)$ or $(2, 1, 1)$.

Free Objects

Let F be an algebra in a class $\mathcal{C}$ of algebras.

Definition

F is **free** in $\mathcal{C}$ if there is a subset $\Sigma \subseteq F$ such that every function from Σ to an algebra $M \in \mathcal{C}$ extends uniquely to a morphism from F to M.

Definition

The cardinality of Σ (which determines F) is a (usually the) **rank** of F.

Example

- Free semigroups
- Free groups
- Free bands
- Free inverse semigroups
- ...

Free Adequate Semigroups

Fact

The class of left adequate semigroups forms a **quasivariety** *of* $(2, 1)$*-algebras defined by:*

- $(xy)z = x(yz)$ *(associativity);*
- $e^2 = e, f^2 = f \implies ef = fe$ *(idempotents commute);*
- $x^+ = (x^+)^+$;
- $x^+x^+ = x^+$;
- $xa = ya \implies xa^+ = ya^+$;
- $xa^+ = ya^+ \implies xa = ya$.

Similarly for right adequate and adequate semigroups.

Corollary

There is a free left/right/two-sided adequate semigroup of every rank.

Corollary

There is a free left/right/two-sided adequate semigroup of every rank.

Question

What is it?

Back to Inverse Semigroups

For the free inverse semigroup, we have the **Munn representation**. *This relies heavily on the* **type A** *identities*

$$ae = (ae)^+ a \text{ and } ea = a(ea)^*$$

and applies in other contexts where these hold.

Question

What happens without these identities?

Corollary

There is a free left/right/two-sided adequate semigroup of every rank.

Question

What is it?

The Story So Far

Branco, Gomes and Gould have recently studied free left and right adequate semigroups from a structural perspective, as part of their theory of **proper** *adequate semigroups.*

Our Aim

A **geometric approach** *(like Munn's) for the both the one-sided and two-sided cases.*

Let Σ be a set (e.g. an alphabet).

Definition

A Σ-**tree** is a directed tree with

- at least one vertex and edge
- each edge labelled by an element of Σ;
- a distinguished **start** vertex;
- a distinguished **end** vertex;
- an undirected path between every pair of vertices;
- a (perhaps empty) directed path from the start to the end.

Definition

A Σ-tree is called **idempotent** if its start and end vertices coincide.

Definition

A **base tree** is a Σ-tree with a single edge and with distinct start and end vertices.

Morphisms

Definition

A **morphism** $\sigma : X \rightarrow Y$ of Σ-trees is a map which

- takes edges to edges;
- takes vertices to vertices;
- preserves incidence;
- preserves edge labels;
- takes the start vertex to the start vertex;
- takes the end vertex to the end vertex.

Definition

$UT(\Sigma)$ is the set of **isomorphism types** of Σ-trees.

Convention

We identify the isomorphism type of a base tree with the label of its edge, so $\Sigma \subseteq UT(\Sigma)$.

Algebra on Trees

Definition

Let $X, Y \in UT(\Sigma)$. Then

- $X \times Y$ is obtained by glueing the end vertex of X to the start vertex of Y.
- $X^{(+)}$ is obtained by moving the end vertex of X to the start vertex.
- $X^{(*)}$ is obtained by moving the start vertex of X to the end vertex.

No folding! (Yet.)

Fact

$UT(\Sigma)$ forms a semigroup under $\times$.

Warning

Idempotent trees are not idempotent! (Yet.)

Retracts

Definition

A **retract** of a Σ-tree is an idempotent morphism from X to X.

Definition

A Σ-tree is called **pruned** if it admits no (non-identity) retracts.

Exercise

Let X be a Σ-tree. Then there is a unique (up to isomorphism) Σ-tree which is the image of a retract of X.

Definition

The (isomorphism type of the) unique pruned image of a retract of X is denoted $\overline{X}$.

Algebra on Pruned Trees

Definition

$T(\Sigma)$ is the set of isomorphism types of **pruned** Σ-trees.

Definition

We define operations on $T(\Sigma)$ by

- $XY = \overline{X \times Y}$;
- $X^+ = \overline{X^{(+)}}$;
- $X^* = \overline{X^{(*)}}$;

for all $X, Y \in T(\Sigma)$.

Theorem

The map $X \mapsto \overline{X}$ is a surjective $(2, 1, 1)$-morphism from $UT(\Sigma)$ to $T(\Sigma)$.

The Free Adequate Semigroup Revisited

Theorem

$T(\Sigma)$ is the free adequate semigroup on Σ.

Left adequate semigroups

Definition

A Σ-tree X is **left adequate** if every edge is orientated away from the start vertex (or equivalently, if there is a path from the start vertex to every vertex).

Definition

$LT(\Sigma)$ with pruned operations is the set of isomorphism types of pruned left adequate Σ-trees.

Theorem

$LT(\Sigma)$ is the free left adequate semigroup on Σ.

Corollary

Any $(2, 1)$-identity which holds in every adequate semigroup also holds every left/right adequate semigroup.

Monoids

Remark

If we admit the **trivial Σ-tree** *with one vertex and no edges, then we obtain the free left/right/two-sided adequate* **monoid***.*

Some Elementary Corollaries

Corollary

The word problem for a finitely generated free left/right/two-sided adequate semigroup is decidable

Question

What is its complexity?

Corollary

One can decide effectively whether a given identity holds in all left/right/two-sided adequate semigroups.

Corollary

No non-trivial free left/right/two-sided adequate semigroup is finitely generated as a semigroup.

Corollary

Every free adequate left/right/two-sided semigroup is $\mathcal{J}$-trivial (as a semigroup).

Inverse Semigroups as Adequate Semigroups

Remark

We can develop an analogous theory in which we

- *replace* **retracts** *with* **morphisms***; and*
- **don't** *require a directed path from the start to the end.*

This gives the Munn representation of the free **inverse** *semigroup. (Isomorphism types of morphism-free Σ-trees are in 1-1 correspondence with Munn trees.)*

Fact

- *There is a natural morphism from the free adequate semigroup to the free inverse semigroup, taking x^+ to xx^{-1} and x^* to $x^{-1}x$.*
- *This can be interpreted as a* **folding** *operation on trees.*
- *Likewise the morphism from the free adequate semigroup* **onto** *the free ample semigroup.*

Residual Finiteness Properties

Definition

A function $f : S \to T$ **separates** $X \subseteq S$ if $x \neq y \implies f(x) \neq f(y)$ for all $x, y \in X$.

Definition

An algebra is **[fully] residually finite** if every pair [finite set] of elements is separated by a morphism to a finite algebra.

Remark

Let F be a free algebra of rank $\aleph_0$ in a class $\mathcal{C}$ of algebras.

- *Pairs of elements in F which cannot be separated in finite quotients correspond to identities which are satisfied in all finite algebras in $\mathcal{C}$, but* **not** *in all infinite algebras.*
- *So F is residually finite $\iff$ every identity satisfied by all finite algebras in $\mathcal{C}$ is also satisfied by all infinite $\mathcal{C}$-algebras.*

Residual Finiteness Properties

Theorem

Free left/right adequate semigroups are (fully) residually finite **as adequate** $(2, 1)$**-algebras**.

Theorem

Every finite subset of a free left/right adequate semigroup is separated by a **Rees quotient** *("fully Rees-residually finite").*

Fact

Finite subsets of free adequate semigroups are **not** *separable by Rees quotients. (There are elements which do* **not** *lie outside a cofinite ideal.)*

Question

Are free adequate semigroups residually finite?