Automaton semigroups

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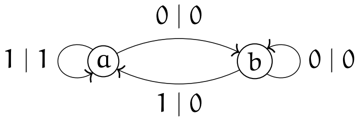
Automata

An automaton $\mathcal{A}$ is a triple (Q, B, δ), where

- Q is a finite set of states
- B is a finite alphabet
- δ is a transformation of the set $Q \times B$

If $(q, b)\delta = (r, c)$ then if $\mathcal{A}$ is in state q, it can read b and move to state r and output c.



Input and output of sequences

If $\mathcal{A}$ starts in state q_0 and reads

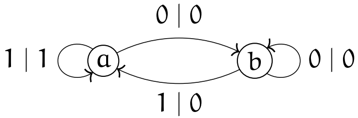
$$\alpha_1\alpha_2\ldots\alpha_n \in B^*,$$

it outputs

$$\beta_1\beta_2\ldots\beta_n,$$

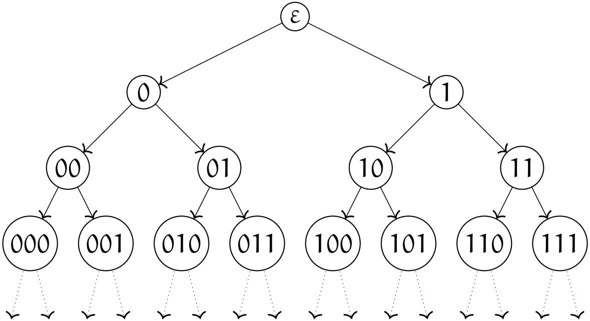
where $(q_{i-1}, \alpha_i)\delta = (q_i, \beta_i)$ for states $q_0, \ldots, q_n \in Q$.
This defines an action of Q on B^*, with

$$\alpha_1\alpha_2\ldots\alpha_n \cdot q = \beta_1\beta_2\ldots\beta_n.$$



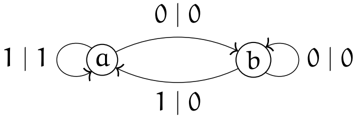
$0110 \cdot a = 0110 \cdot a = 0(110 \cdot b)$ $0110 \cdot a = 00(10 \cdot b)$
$0110 \cdot a = 001(0 \cdot a)$ $0110 \cdot a = 0010$

$\mathbb{B}^*$ as a tree



Action on B^*

Each $q \in Q$ acts on B^*, so there is a map $\phi : Q \to \operatorname{End} B^*$, which extends to a homomorphism $\phi : Q^+ \to \operatorname{End} B^*$.



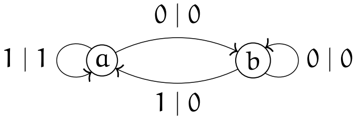
$1110 \cdot babb = 1110 \cdot babb = 0110 \cdot abb$ $1110 \cdot babb = 0010 \cdot bb$
$1110 \cdot babb = 0000 \cdot b$ $1110 \cdot babb = 0000$

The semigroup from the action

The image of $\phi : Q^+ \to \operatorname{End} B^*$, denoted $\Sigma(\mathcal{A})$, is the automaton semigroup defined by $\mathcal{A}$.

Identify each $q \in Q$ with its image under ϕ, so that Q becomes a generating set for $\Sigma(\mathcal{A})$.

The semigroup from the action

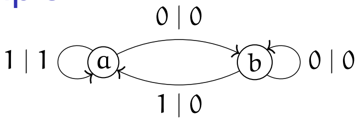


$\alpha \cdot b$ must begin with a 0. Write $\alpha \cdot b = 0\beta$. Then

$$(0\beta) \cdot a = 0(\beta \cdot b) \quad \text{and} \quad (0\beta) \cdot b = 0(\beta \cdot b).$$

So $\alpha \cdot ba = \alpha \cdot b^2$ for any $\alpha \in B^*$; thus $ba = b^2$.

Back to the example



Every element of $\Sigma(\mathcal{A})$ can be written as $a^i b^j$. Now,

$$0^k 1^\omega \cdot a = 0^k 1^\omega \cdot b = 0^{k+1} 1^\omega .$$

Thus, for $i, j \in \mathbb{N}^0$,

$$01^\omega \cdot a^i b^j = 0^{i+j+1} 1^\omega .$$

Furthermore, for $n > j$,

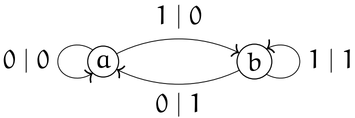
$$1^n 0^\omega \cdot a = 1^n 0^\omega ,$$

and hence

$$1^n 0^\omega \cdot a^i b^j = 1^n 0^\omega \cdot b^j = 0^j 1^{n-j} 0^\omega .$$

So if $a^i b^j = a^k b^l$, then $i + j + 1 = k + l + 1$ and $j = l$, whence $i = k$. The semigroup is therefore presented by $\langle a, b \mid (ba, b^2) \rangle$.

Another example



1. Having read symbol 0, the automaton enters a; on reading 1, it enters b.
2. Leaving state a the output is 0; leaving b the output is 1.

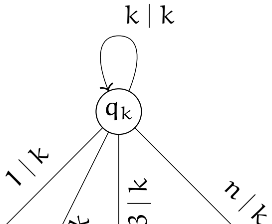
So the automaton remembers the last read symbol and output its when the next symbol is read.
So a acts by shifting right and inserting 0 and b shifts right and inserts 1.

Thus the word $w \in Q^+$ is determined by the common prefix of $\alpha \cdot w$ for long $\alpha \in B^*$. So $\Sigma(\mathcal{A})$ is free with basis Q.

A generalization

Theorem

Every free semigroup of rank at least 2 is an automaton semigroup.



Wreath recursions

The endomorphism semigroup of B^* decomposes as:

$$\mathrm{End}\, B^* = \mathrm{End}\, B^* \wr \mathcal{T}_B.$$

That is,

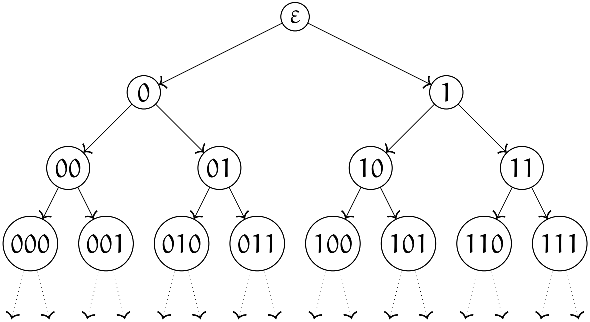
$$\mathrm{End}\, B^* = \big(\underbrace{\mathrm{End}\, B^* \times \ldots \times \mathrm{End}\, B^*}_{|B| \text{ times}}\big) \rtimes \mathcal{T}_B$$

So if $q \in \mathrm{End}\, B^*$, then

$$q = (x_1, \ldots, x_{|B|})\tau$$

for $x_i \in \mathrm{End}\, B^*$ and $\tau \in \mathcal{T}_B$. This is called the wreath recursion associated to q.

$\mathbb{B}^*$ as a tree



$p = (q, r)\tau$

Wreath recursions for automaton semigroups

Define $\tau_q : B \to B$ and $\pi_q : B \to Q$ such that $(q, b)\delta = (b\pi_q, b\tau_q)$.

The wreath recursion associated to $q \in Q$ is

$$q = (1\pi_q, 2\pi_q, \ldots, n\pi_q)\tau_q.$$

Calculating with wreath recursions

Suppose

$$p = (x_0, x_1, \cdots, x_{d-1})\tau$$

and

$$q = (y_0, y_1, \cdots, y_{d-1})\rho$$

Then

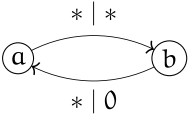
$$pq = (x_0 y_{0\tau}, x_1 y_{1\tau}, \cdots, x_{d-1} y_{(d-1)\tau})\tau\rho.$$

For example, let $a = (b, c)\lambda$ and $d = (e, f)\rho$, where $x\lambda = 0$ and $x\rho = 1$. Then

$$ad = (be, ce)\rho, \qquad da = (ec, fc)\lambda.$$

Example of using wreath recursions

Automaton acting on $\{0,1\}^*$:

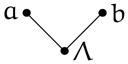


Wreath recursions: $a = \mathrm{id}(b,b)$, $b = \lambda(a,a)$.

$$a^2 = (b^2, b^2)\,\mathrm{id} = a$$
$$b^2 = (a^2, a^2)\lambda = b$$
$$ab = (ba, ba)\lambda = \Lambda$$
$$ba = (ab, ab)\lambda = \Lambda$$

($\alpha\Lambda = 0^{|\alpha|}$ for any α in$\{0,1\}^*$.) Also $0^k \cdot a = 0^k \cdot b = 0^k$
Hence $\Lambda a = \Lambda b = a\Lambda = b\Lambda = \Lambda\Lambda = \Lambda$

So we have a three-element semilattice:



Word problem

Let $u, v \in Q^+$.

- Compute the wreath recursions for u and v:

$$u = (w_1^{(u)}, \dots, w_n^{(u)})\tau_u \text{ and } v = (w_1^{(v)}, \dots, w_n^{(v)})\tau_v.$$

- Check whether $\tau_u = \tau_v$.
- Check whether $w_i^{(u)} = w_i^{(v)}$ for each $i \in B$.

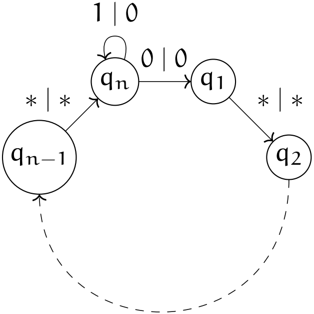
The algorithm terminates because $|w_i^{(u)}| = |u|$ and $|w_i^{(v)}| = |v|$.

Free commutative semigroups

Theorem

Every free commutative semigroup of rank at least 2 is an automaton semigroup.

Free commutative semigroups



$q_i = \mathrm{id}(q_{i+1}, q_{i+1})$ for $i = 1, \ldots, n-1$, and $q_n = \lambda(q_1, q_n)$

$$
\begin{aligned}
q_i q_j &= \mathrm{id}(q_{i+1} q_{j+1}, q_{i+1} q_{j+1}), \\
q_j q_i &= \mathrm{id}(q_{j+1} q_{i+1}, q_{j+1} q_{i+1}) \qquad \text{for } i, j = 1, \ldots, n-1, \\
q_i q_n &= \lambda(q_{i+1} q_1, q_{i+1} q_n), \\
q_n q_i &= \lambda(q_1 q_{i+1}, q_n q_{i+1}) \qquad \text{for } i = 1, \ldots, n-1.
\end{aligned}
$$

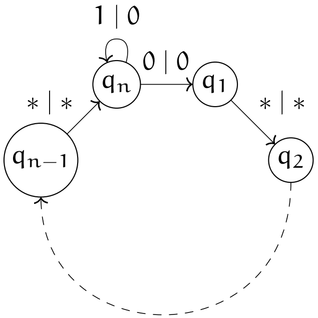
Free commutative semigroups

So every element of the semigroup is a product $q_1^{k_1} \cdots q_n^{k_n}$.

Need to show that every element has a unique such expression.

The aim is to show that the action of $q_1^{k_1} \cdots q_n^{k_n}$ determines each k_i.

Free commutative semigroups



Let $\alpha_i = 0^{n-i}10^{i-1}$. Let $\zeta = 0^n$.

$$
\begin{aligned}
(\alpha_i \beta) \cdot q_i &= \zeta(\beta \cdot q_{i-1}) && \text{for } i = 1, \dots, n, \\
(\alpha_i \beta) \cdot q_j &= \alpha_i(\beta \cdot q_j) && \text{for } i, j = 1, \dots, n \text{ with } i \neq j, \\
(\zeta \beta) \cdot q_i &= \zeta(\beta \cdot q_i) && \text{for } i = 1, \dots, n.
\end{aligned}
$$

Consequently,

$$\alpha_i^k \cdot w = \zeta^{|w|_{q_i}} \alpha_i^{k-|w|_{q_i}}.$$

Automaton groups

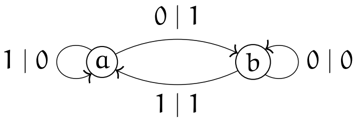
$\mathcal{A} = (Q, B, \delta)$ is invertible if the action of every state q on B^* is a bijection.

$\mathcal{A}$ is invertible if and only if each τ_q is a bijection.

If $\mathcal{A}$ is invertible, there is a natural action of q^{-1} on B^* and so a natural map $\phi : (Q \cup Q^{-1})^+ \to \operatorname{Aut} B^*$. The automaton group $\Gamma(\mathcal{A})$ is the image of ϕ.

Automaton groups

The lamplighter group $\mathbb{Z}_2 \operatorname{wr} \mathbb{Z}$ is $\Gamma(\mathcal{A})$, where $\mathcal{A}$ is:



Examples of 2-state 2-symbol automaton groups

Theorem (Grigorchuk et al.)

Let $\mathcal{A}$ be a 2-state 2-symbol invertible automaton. Then $\Gamma(\mathcal{A})$ is one of:

- *the trivial group*
- $\mathbb{Z}_2$
- $\mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}$
- *the infinite dihedral group*
- *the lamplighter group* $\mathbb{Z}_2 \operatorname{wr} \mathbb{Z}$

Examples of 2-state 2-symbol automaton semigroups

Let $\mathcal{A}$ be a 2-state 2-symbol automaton. Then $\Sigma(\mathcal{A})$ may be:

- Trivial
- 2-element chain
- 2-element left zero semigroup
- $\langle a, b \mid (a^2, a), (b^2, b), (ba, b) \rangle$ (3 elements)
- 3-element non-chain semilattice
- $\langle a, b, 0 \mid (a^3, a^2), (ab, ba), (ab, 0) \rangle$ (4 elements)
- $\mathbb{N} \cup \{0\}$
- Free product of two trivial semigroups
- Free commutative semigroup of rank 2
- Free semigroup of rank 2
- $\langle a, b \mid (ba, b^2) \rangle$

Bicyclic monoid

Proposition

The bicyclic monoid is not an automaton semigroup.

Proof.

Suppose $\langle b, c \mid (bc, \varepsilon) \rangle$ is $\Sigma(\mathcal{A})$, where $\mathcal{A} = (Q, B, \delta)$.

So bc acts identically on B^* and cb acts non-identically.

That is, bc acts identically on B^n for some n and cb acts non-identically.

So b acts injectively and so bijectively on B^n.

Thus c and b are inverse mappings on B^n and so cb acts identically on B^n.

□

Basic properties of automaton semigroups

Proposition

Every automaton semigroup is residually finite.

Proposition

Every automaton semigroup is hopfian.

Adjoining zeroes and identities

Proposition

If S is an automaton semigroup, then so is S^0.

Proposition

If S is an automaton semigroup, then so is S^1.

$\mathbb{N} \cup \{0\}$ is an automaton semigroup but $\mathbb{N}$ is not.

Direct products

Proposition

Let S *and* T *be automaton semigroups. Then* $S \times T$ *is an automaton semigroup if and only if it is finitely generated.*

Cayley automata

Let S be a finite semigroup. The Cayley automaton $\mathcal{C}(S)$ is (S, S, δ), where $(s, t)\delta = (st, st)$:

- $\mathcal{C}(S)$ acts on S^*.
- pq is ambiguous – a product or a sequence of two symbols.
- Henceforth use overlines to distinguish: $\overline{p}\,\overline{q}$ or $\overline{pq}$.

An example

Suppose L is a finite left zero semigroup.
When $\mathcal{C}(L)$ is in state $\overline{q}$ and reads $\overline{x}$, it moves to state $\overline{qx} = \overline{q}$ and outputs $\overline{qx} = \overline{q}$.

$\overline{\alpha_1}\,\overline{\alpha_2}\ldots\overline{\alpha_n}\cdot\overline{q} = \overline{\alpha_1}\,\overline{\alpha_2}\ldots\overline{\alpha_n}\cdot\overline{q} = \overline{q}\,(\overline{\alpha_2}\ldots\overline{\alpha_n}\cdot\overline{q})\overline{\alpha_1}\,\overline{\alpha_2}\ldots\overline{\alpha_n}\cdot\overline{q} = \overline{q}\,\overline{q}(\ldots\overline{\alpha_n}\cdot\overline{q})\overline{\alpha_1}\,\overline{\alpha_2}\ldots\overline{\alpha_n}\cdot\overline{q} = \overline{q}\,\overline{q}\ldots\overline{q}$

So $\alpha\cdot\overline{q} = \overline{q}^{|\alpha|}$, and $\alpha\cdot\overline{q}\,\overline{r} = \overline{r}^{|\alpha|}$.
$\Sigma(\mathcal{C}(L))$ is a right zero semigroup of cardinality L.

A theorem and a generalization

Theorem (Silva & Steinberg)

If G is a finite non-trivial group, then $\Sigma(\mathcal{C}(G))$ is a free semigroup of rank $|G|$.

Theorem

If S is a finite Clifford semigroup with all maximal subgroups non-trivial, then $\Sigma(\mathcal{C}(S))$ is a strong semilattice of free semigroups.

Characterization of groups arising from Cayley automata

Theorem (Maltcev)

The following are equivalent:

1. $\Sigma(\mathcal{C}(S))$ *is a group*
2. $\Sigma(\mathcal{C}(S))$ *is trivial*
3. S *is an inflation of right zero semigroups by null semigroups*

Question

Is $\Sigma(\mathcal{C}(S))$ always aperiodic (has trivial $\mathcal{H}$-classes)?

Characterizing finite Cayley automata semigroups

Theorem

$\Sigma(\mathcal{C}(S))$ is finite if and only if S is aperiodic.

There are three proofs of the ‘if’ part of this:

- Detailed calculations with wreath recursions (Maltcev).
- By considering a restricted action on sequences of elements of an ideal of S (Mintz)
- Combinatorial arguments on the action on sequences (C).

Corollary

If $\Sigma(\mathcal{C}(S))$ is infinite, it contains a free semigroup of rank 2.

Corollary

$\mathbb{N} \cup \{0\}$ is not a Cayley automaton semigroup.

$\Sigma(\mathcal{C}(S)) \simeq S$?

Proposition

If S is a semilattice, then $\Sigma(\mathcal{C}(S)) \simeq S$.

Proposition

If S is an $\mathrm{I} \times \mathrm{I}$ rectangular band, then $\Sigma(\mathcal{C}(S^1)) \simeq S^1$.

Conjecture

The semigroups S with $\Sigma(\mathcal{C}(S)) \simeq S$ are precisely the finite bands wherein every rectangular band is 'square' and each maximal $\mathcal{D}$-class is a singleton.

Constructions on the underlying semigroup

Proposition

$\Sigma(\mathcal{C}(S^0)) \simeq \Sigma(\mathcal{C}(S))^0$.

Unfortunately, $\Sigma(\mathcal{C}(S^1))$ is not, in general, isomorphic to $\Sigma(\mathcal{C}(S))^1$.

Proposition

$\Sigma(\mathcal{C}(S \cup_0 T)) \simeq \Sigma(\mathcal{C}(S)) \cup_0 \Sigma(\mathcal{C}(T))$.

Open problems

Problem

Decision problems for automaton semigroups: given the automaton $\mathcal{A}$ as input, what properties of $\Sigma(\mathcal{A})$ can be decided?

Problem

Consider more general automata than can output zero or multiple symbols for each input symbol (i.e. $\delta : Q \times B \to Q \times B^*$).