Congruences of End$F_n(G)$

FFB + NR

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Outline of this talk:

- Definitions and notation.
- Some structure of End $F_n(G)$.
- Congruences of rank one.
- Congruences of higher rank.

Let X be a non-empty set, G be a group such that $G \curvearrowright X$. We say X is a left G-act, and write ${}_G X$.

Definition (Free act)

A generating set U of ${}_G X$ is a *basis* if every $x \in {}_G X$ can be uniquely presented in the form $x = gu$ for some $u \in U$, $g \in G$. That is:

$$x = g_1 u_1 = g_2 u_2 \iff g_1 = g_2 \quad \text{and} \quad u_1 = u_2$$

If an act ${}_G X$ has a basis U, then it is called the *free rank* $|U|$ *act*, and we write ${}_G X = F_{|U|}(G)$ to denote this.

Free rank n G-act

Let G be a group, and let

$$F_n(G) = \bigcup_{i=1}^{n} Gx_i$$

be the rank n free left G-act.
$F_n(G)$ consists of the set of formal symbols $\{gx_i : g \in G, 1 \leq i \leq n\}$. For any $g, h \in G$ and $1 \leq i, j \leq n$:

$$gx_i = hx_j \iff g = h \quad \text{and} \quad i = j$$

The action of G is given by $g(hx_i) = (gh)x_i$.

Endomorphisms of $F_n(G)$

Definition (Act Endomorphism)

Let G act on a set A. Then $\phi : A \longrightarrow A$ is an *act endomorphism* if

$$(ga)\phi = g(a\phi) \quad \forall g \in G,\ a \in A$$

Write $\mathrm{End}F_n(G)$ for the collection of all endomorphisms of $F_n(G)$, with composition of maps as its binary operation.

Each $\alpha \in \mathrm{End} F_n(G)$ is determined completely by its act on the free generators $\{x_i \,:\, i \in [1,n]\}$, therefore we can write:

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ \omega_1^\alpha x_{1\bar{\alpha}} & \omega_2^\alpha x_{2\bar{\alpha}} & \dots & \omega_n^\alpha x_{n\bar{\alpha}} \end{pmatrix}$$

for a map $\bar{\alpha} \in T_n$ and an element $\alpha_G = (\omega_1^\alpha, \omega_2^\alpha, \dots, \omega_n^\alpha) \in G^n$.

Fact

The function

$$\begin{aligned} \psi : \mathrm{End} F_n(G) &\longrightarrow G \wr T_n \\ \alpha &\mapsto (\alpha_G, \bar{\alpha}) \end{aligned}$$

is an isomorphism.

Image, Rank, Kernel

Image And Rank

Let $\alpha \in \operatorname{End} F_n(G)$, then:

$$\operatorname{im}(\alpha) = \bigcup_{i \in \operatorname{im}(\bar{\alpha})} Gx_i, \quad \operatorname{rank}(\alpha) = \operatorname{rank}(\bar{\alpha})$$

Write D_m for the $\mathcal{D}$–class of elements of rank m.

Kernel

Let $\alpha, \beta \in D_r$. Then $\ker(\alpha) = \ker(\beta)$ if and only if $\ker(\bar{\alpha}) = \ker(\bar{\beta}) = \{B_1, \ldots, B_r\}$ and for any $j \in \{1, \ldots, r\}$ there exists $q_{j,\alpha,\beta} \in G$ such that for any $k \in B_j$, we have $\omega_k^{\alpha} q_{j,\alpha,\beta} = \omega_k^{\beta}$

Example

Consider $\alpha, \beta, \gamma \in \operatorname{End} F_3(C_2)$, $C_2 = \langle a \rangle$.

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_3 \\ ax_1 & x_1 & x_2 \end{pmatrix} \quad \beta = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & ax_1 & x_2 \end{pmatrix}$$

$$q_{1,\alpha,\beta} = q_{2,\alpha,\beta} = a, \quad q_{3,\alpha,\beta} = 1$$

Hence $\ker(\alpha) = \ker(\beta)$. But:

$$\gamma = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_1 & x_2 \end{pmatrix}$$

$$q_{1,\alpha,\gamma} = a \neq 1 = q_{2,\alpha,\gamma}, \quad q_{3,\alpha,\gamma} = 1$$

Hence $\ker(\alpha) \neq \ker(\gamma)$.

Structure of End$F_n(G)$

Green's Relations on End$F_n(G)$

For any $\alpha, \beta \in \mathrm{End}F_n(G)$:

- $\alpha\mathcal{L}\beta \iff \mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
- $\alpha\mathcal{R}\beta \iff \ker(\alpha) = \ker(\beta)$,
- $\alpha\mathcal{D}\beta \iff \mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$,

Definition (Rank of a congruence)

Let ρ be a congruence of $\mathrm{End}F_n(G)$, then the *rank* of ρ, written $\mathrm{rank}(\rho)$ is:

$$\mathrm{rank}(\rho) = \max\{\, \mathrm{rank}(f) \mid \exists\, g \neq f \text{ such that } (g, f) \in \rho\}$$

Remark

Let H_m be an $\mathcal{H}$–group class of D_m, then:

$$H_m \cong G \wr \mathrm{Sym}(m)$$

In particular,

$$H_1 \cong G \wr \mathrm{Sym}(1) \cong G$$

Fact

Let S and T be semigroups, $f : S \to T$ be a homomorphism, and ρ be a congruence on T. Then

$$f^{-1}(\rho) = \{(\alpha, \beta) \in S^2 \mid (\alpha f, \beta f) \in \rho\}$$

is a congruence on S.

Fact

The map $f : \operatorname{End} F_n(G) \to T_n$ defined by

$$\alpha \mapsto \bar{\alpha}$$

is a homomorphism.

For the next few slides, we restrict our attention to congruences of rank one. That is, congruences whose non-trivial classes contain only elements of D_1.

Congruences in $\mathcal{H}$

Fact

Let $\alpha, \beta \in \mathrm{End} F_n(G)$ be such that $(\alpha, \beta) \in D_1 \times D_1$, and $\alpha \mathcal{H} \beta$. Let N be the normal subgroup of G generated by $q_{\alpha,\beta}$. Then:

$$(\alpha, \beta)^\sharp = \{(\gamma, \delta) \in D_1 \times D_1 \mid \gamma \mathcal{H} \delta,\ q_{\delta,\gamma} \in N\} \cup \Delta$$

Congruences in $\mathcal{R}\backslash\mathcal{L}$

Fact

Let $\alpha, \beta \in \mathrm{End}F_n(G)$ be such that $(\alpha, \beta) \in D_1 \times D_1$, and $(\alpha, \beta) \in \mathcal{R}\backslash\mathcal{L}$. Then:

$$(\alpha, \beta)^\sharp = \{(\gamma, \delta) \in D_1 \times D_1 \mid \gamma\mathcal{R}\delta\} \cup \Delta$$

Congruences in $\mathcal{L}\backslash\mathcal{R}$

Fact

Let $\alpha, \beta \in \operatorname{End} F_n(G)$ be elements of rank one such that $(\alpha, \beta) \in \mathcal{L}\backslash\mathcal{R}$. Let M be the normal subgroup of G generated by $\{(q_{1,\alpha,\beta})^{-1}q_{1,\alpha,\beta}, \ldots, (q_{1,\alpha,\beta})^{-1}q_{n,\alpha,\beta}\}$, and N be the normal subgroup of G generated by $\{q_{1,\alpha,\beta}, \ldots, q_{n,\alpha,\beta}\}$. Then:

$$(\alpha, \beta)^\sharp = \left\{(\gamma, \delta) \in D_1^2 \mid \gamma\mathcal{L}\delta,\ q_{k,\gamma,\delta} \in Mt \text{ for some } t \in N, k \in \{1, \ldots, n\}\right\} \cup \Delta.$$

Example

Let $C_4 = \langle a \rangle$, be the cyclic group of order four. Consider End $F_3(C_4)$. Let $\alpha, \beta \in$ End $F_3(C_4)$ be:

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_3 \\ ax_1 & a^2x_1 & a^3x_1 \end{pmatrix},$$
$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 \\ a^2x_1 & ax_1 & a^2x_1 \end{pmatrix}.$$

Then:

$$q_{k,\alpha,\beta} = \begin{cases} a & \text{if } k = 1 \\ a^3 & \text{if } k \in \{2,3\} \end{cases},$$
$$(q_{1,\alpha,\beta})^{-1} q_{k,\alpha,\beta} = \begin{cases} 1 & \text{if } k = 1 \\ a^2 & \text{if } k \in \{2,3\} \end{cases}.$$

We then have that

$$\langle \bigcup_{k=1}^{n} \mathscr{C}((q_{j,\alpha,\beta})^{-1} q_{k,\alpha,\beta}) \rangle = \langle a^2 \rangle = C_2 \neq C_4 = \langle a \rangle = \langle \bigcup_{k=1}^{n} \mathscr{C}(q_{k,\alpha,\beta}) \rangle$$

Congruences not in $\mathcal{R} \cup \mathcal{L}$

Fact

Let $\alpha, \beta \in \mathrm{End} F_n(G)$ be such that $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta) = 1$, $(\alpha, \beta) \notin \mathcal{L} \cup \mathcal{R}$. Define $\gamma \in \mathrm{End} F_n(G)$ by:

$$x_k \gamma = \omega_k^{\beta} x_{1\bar{\alpha}}$$

for $k \in \{1, \ldots, n\}$. Then:

$$(\alpha, \beta)^{\sharp} = (\alpha, \gamma)^{\sharp} \vee (\gamma, \beta)^{\sharp}$$

What about congruences of higher rank?

Definition

Let ρ be a congruence on $S = \operatorname{End} F_n(G)$. We say that ρ is of:

1. Δ-type if $\rho \subseteq \left\{(\alpha, \beta) \in S^2 \mid \bar{\alpha} = \bar{\beta}\right\}$;
2. Ideal type if $D_1^2 \subseteq \rho$;
3. Complementary type if there exist $\alpha, \beta \in S$ such that $\bar{\alpha} \neq \bar{\beta}$ and $(\alpha, \beta) \in \rho$, but $D_1^2 \not\subseteq \rho$.

If ρ is of ideal type, then it has a unique ideal congruence class $\mathcal{I}_\rho = \mathcal{I}_r$, and it contains all elements of rank at most r.

Fact

Let ρ be a congruence on End $F_n(G)$ different from the equality congruence. Let $(\alpha, \beta) \in \rho \backslash \Delta$, and let $gx_m \in F_n(G)$ be such that $(gx_m)\alpha \neq (gx_m)\beta$. Let N be the normal subgroup of G generated by $q_{m,\alpha,\beta}$. Then:

$$\left\{(\gamma, \delta) \in D_1^2 \;\middle|\; \gamma \mathcal{H} \delta,\ q_{\gamma,\delta} \in N\right\} \subseteq \rho.$$

Furthermore, if $m\bar{\alpha} \neq m\bar{\beta}$, then:

$$\left\{(\gamma, \delta) \in D_1^2 \;\middle|\; \gamma \mathcal{R} \delta\right\} \subseteq \rho\ .$$

Fact

Let ρ be a congruence on $\operatorname{End} F_n(G)$. Suppose $\alpha, \beta \in \operatorname{End} F_n(G)$ are such that $\operatorname{rank}(\beta) < \operatorname{rank}(\alpha) = k$, and $(\alpha, \beta) \in \rho$. Then ρ is of ideal type, and $\mathcal{I}_k \subseteq \mathcal{I}_\rho$.

Corollary

Let $\alpha, \beta \in \operatorname{End} F_n(G)$ be such that $\operatorname{rank}(\beta) < \operatorname{rank}(\alpha) = k$. Then $(\alpha, \beta)^\sharp = \operatorname{Rees}(k)$.

Fact

Let ρ be a congruence of ideal type. If $\alpha, \beta \in \operatorname{End} F_n(G)$ are such that $(\alpha, \beta) \in \rho$ and $\alpha \notin \mathcal{I}_\rho$, then $\alpha \mathcal{L} \beta$ and $\ker(\bar{\alpha}) = \ker(\bar{\beta})$.

Corollary

Let ρ be a congruence of complementary type. If $\alpha, \beta \in \operatorname{End} F_n(G)$ are such that $(\alpha, \beta) \in \rho$ and $\alpha \notin D_1$, then $\alpha \mathcal{L} \beta$ and $\ker(\bar{\alpha}) = \ker(\bar{\beta})$.

Fact

Let ρ be a congruence of ideal type, $\alpha \in \operatorname{End} F_n(G)$ an element of rank k, $k \geq 2$, such that there exists $\beta \in \operatorname{End} F_n(G)$, $\beta \neq \alpha$, such that $(\alpha, \beta) \in \rho$. Then $\mathcal{I}_{k-1} \subseteq \mathcal{I}_\rho$.

Corollary

Let ρ be a congruence of complementary type. If $\alpha, \beta \in \operatorname{End} F_n(G)$ are such that $(\alpha, \beta) \in \rho$ and $\alpha \notin D_1 \cup D_2$, then $\bar{\alpha} = \bar{\beta}$.

Corollary

Let ρ be a congruence of ideal type, and $\mathcal{I}_\rho = \mathcal{I}_k$. If $\alpha, \beta \in \operatorname{End} F_n(G)$ are such that both are of rank greater than $k+1$ and they are related by ρ, then $\alpha = \beta$.

Corollary

Let τ be a congruence on $\operatorname{End} F_n(G)$ of complementary type, then

$$\tau = \rho \vee \sigma,$$

where ρ is a congruence of rank one or two, and σ is a congruence of Δ-type.