The 0-rook monoid and friends

Florent Hivert – joint work with Joël Gay

LRI / Université Paris Sud 11 / CNRS

NBSAN / June 2018

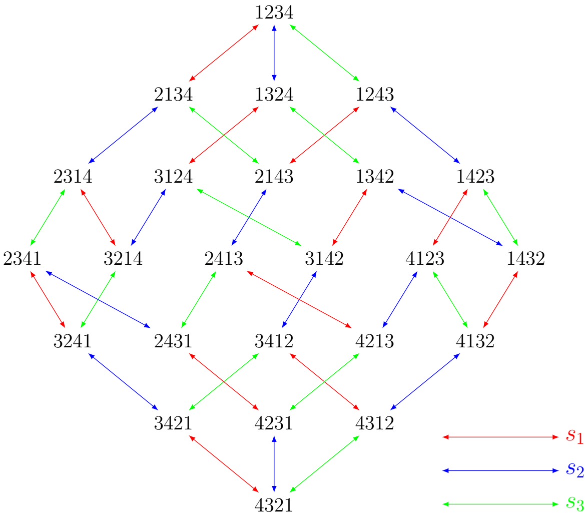
Overview

A strange coincidence between

- Semigroup properties
- Partially ordered set and lattice properties
- Geometric properties

Outline

Background: The right Cayley graph of the symmetric group



Coxeter's presentation of the symmetric group $\mathfrak{S}_n$

$\mathfrak{S}_n$ is generated by the elementary transpositions:

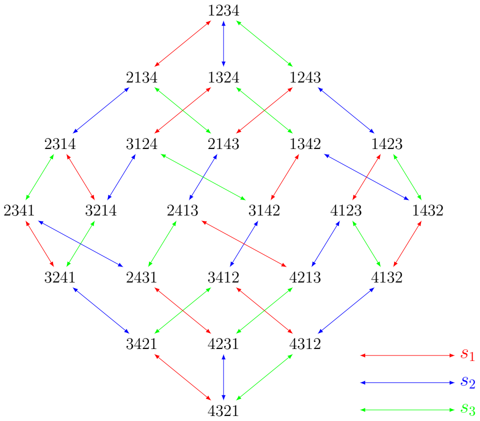
$$s_i := (i, i+1)$$

with relations

$$s_i^2 = \mathrm{Id}$$

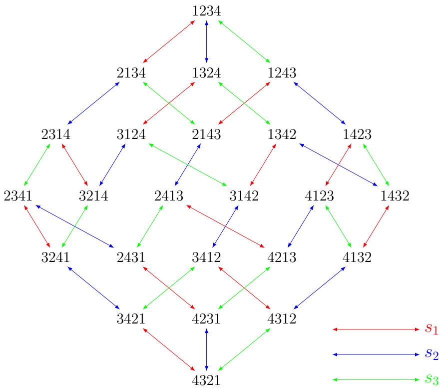
$$s_i s_j = s_j s_i \quad |i-j| \geq 2,$$

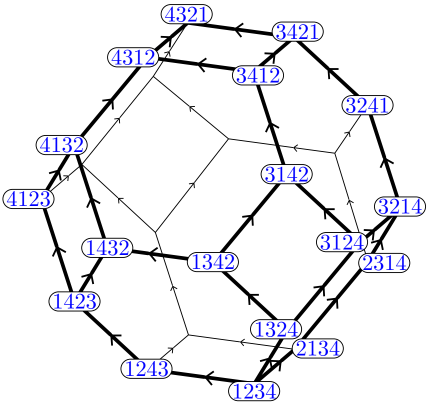
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$



Cayley graph as the skeleton of a polytope

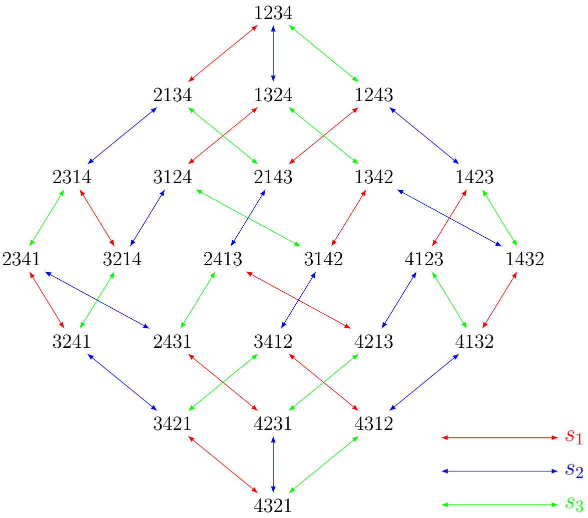
Convex hull of the orbit of $(1, 2, 3, \dots, n)$.
Lives in a $n - 1$ dimensional hyperplane. 3D thanks to Sage, ppl, threejs, jmol

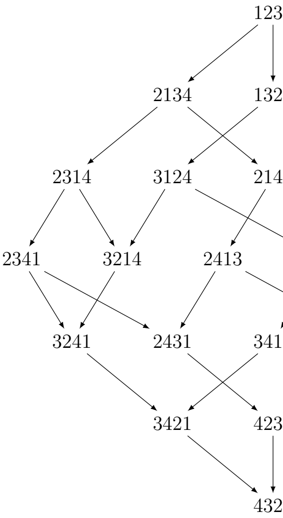




Cayley graph as the Hasse diagram of a lattice

[Guilbaud-Rosenstiehl 1963]



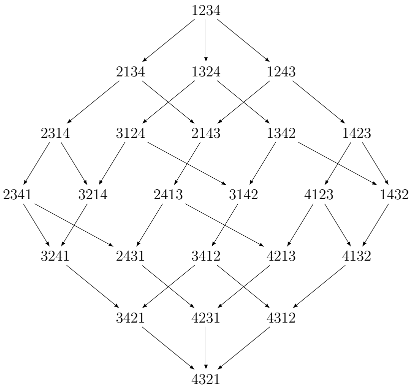


Cayley graph as the Hasse diagram of a lattice

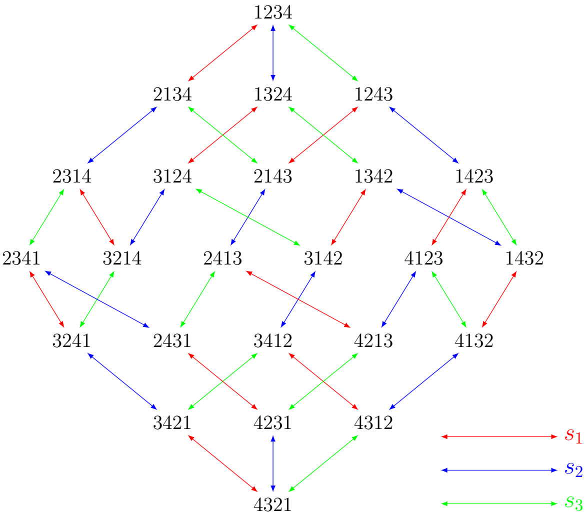
Lattice ≡ partial order with

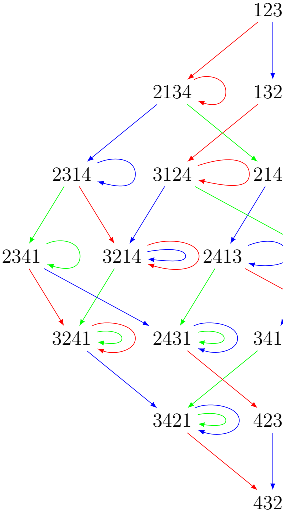
- meet (least upper bound)
- join (greatest lower bounds)

Is there a semigroup interpretation of this partial order ?



Symmetries and projections

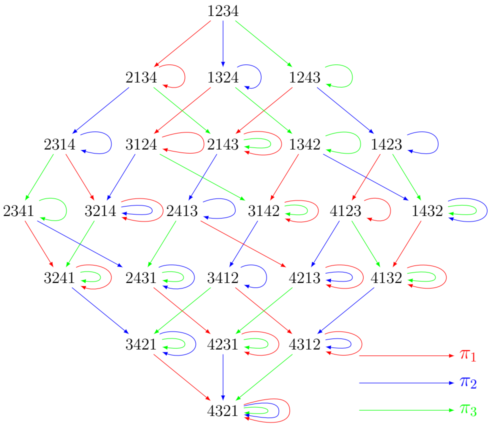




The 0-hecke monoid as a transformation monoid

Transformation monoid generated by the elementary bubble sorting operators π_i

$$31 24 \cdot \pi_1 = 3124$$
$$3124 \cdot \pi_2 = 3214$$
$$3124 \cdot \pi_3 = 3142$$



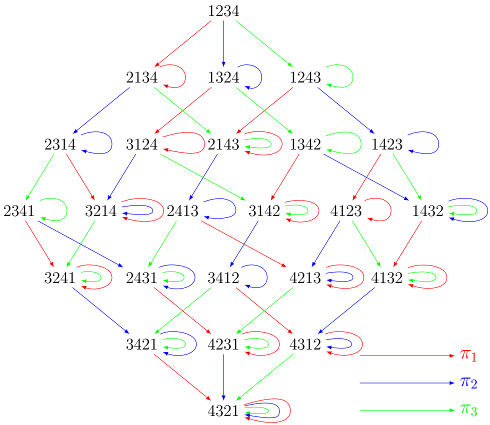
The 0-hecke monoid

The 0-Hecke monoid H_n^0 defined by presentation:

$$\pi_i^2 = \pi_i$$

$$\pi_i\pi_j = \pi_j\pi_i \quad |i-j| \geq 2,$$

$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$$



Matsumoto theorem

Reduced word for a permutation σ :

- decomposition on the s_i of **minimal length**.
- path from Id to σ going **down** in the Cayley graph of $\mathfrak{S}_n$.

Theorem (Matsumoto)

Two reduced words give the same permutation if and only if they can be related using only the braid relations:

$$s_i s_j = s_j s_i \quad |i-j| \geq 2,$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

Consequences of Matsumoto theorem

- if $s_{i_1} \dots s_{i_k} = s_{j_1} \dots s_{j_k} = \sigma$ are two reduced words for σ, then

  $$\pi_{i_1} \dots \pi_{i_k} = \pi_{j_1} \dots \pi_{j_k}$$

  in H_n^0.

- $\pi_\sigma := \pi_{i_1} \dots \pi_{i_k}$ is independent of the chosen reduced word $\sigma = s_{i_1} \dots s_{i_k}$ and therefore well defined.

- $H_n^0 = \{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ in particular $\mathrm{Card}(H_n^0) = n!$

Remarks

- If $s_{i_1} \dots s_{i_k}$ is reduced then

$$\mathrm{Id} \cdot s_{i_1} \dots s_{i_k} = \mathrm{Id} \cdot \pi_{i_1} \dots \pi_{i_k}$$

- π_σ is characterized by $\mathrm{Id} \cdot \pi_\sigma = \sigma$

- The action on permutation is nothing but right multiplication:

$$\pi_\sigma \pi_i = \pi_{(\sigma \cdot \pi_i)}$$

Some more background on H_n^0

- Construction of H_n^0 generalizes to any Coxeter group [Norton 1979, Carter 1981].

- One can interpolate between $\mathbb{C}\mathfrak{S}_n$ and $\mathbb{C}H_n^0$ [Iwahori 1964, Lascoux-Schützenberger 1987]:

$$T_i := q\, s_i + (1-q)(\pi_i - 1)$$

  Iwahori-Hecke algebra $H_n(q)$ with $\mathbb{C}\mathfrak{S}_n \approx H_n(1)$ and $\mathbb{C}H_n^0 \approx H_n(0)$.

H_n^0 and representation theory

- [Demazure 1974] Action of H_n^0 on polynomials via Newton's divided differences:

$$f(\dots x_i, x_{i+1} \dots) \cdot \pi_i = \frac{x_i f(x_i, x_{i+1}) - x_{i+1} f(x_{i+1}, x_i)}{x_i - x_{i+1}}$$

  Factorize Jacobi's symmetrizer (def. of Schur function) $\equiv$ Weyl-character formula: Demazure character formula

- $\mathcal{R}$-trivial and self opposite and thus $\mathcal{J}$-trivial. Allows to analyse its representation theory [Denton-H.-Shilling-Thiéry 2011]

- Representation theory related to the Hopf algebras of quasi-symmetric and non commutative symmetric functions [Krob-Thibon 1997, H. 1999]

From permutations to rooks

Rook Matrix	$\begin{pmatrix} 0 & 0 & 0 & 0 & \text{♖} \\ 0 & 0 & \text{♖} & 0 & 0 \\ 0 & 0 & 0 & \text{♖} & 0 \\ 0 & \text{♖} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \text{♖} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \text{♖} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{♖} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
Rook Vector	0 4 2 3 1	0 3 0 4 1

Rooks = partial permutations of $\{1 \dots n\}$ (in vector $0 =$undefined).

The product of two rook matrices is a rook matrix
The compose of two partial permutation is a partial permutation.
Rook Monoid R_n = submonoid of the rook matrices

$$\mathfrak{S}_n \subset R_n \subset M_n$$

Presentation of the Rook Monoid

Generators: elementary transpositions s_i, deletion π_0.

$$s_1 = \mathbf{21}34, \quad s_2 = 1\mathbf{32}4, \quad s_3 = 12\mathbf{43}, \quad \pi_0 = \mathbf{0}234$$

Right multiplication:

$$\begin{aligned}(r_1 \dots r_n) \cdot s_i &= r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n \\ (r_1 \dots r_n) \cdot \pi_0 &= 0 r_2 \dots r_n.\end{aligned}$$

Example:

$$3610200 \cdot s_1 = \mathbf{63}10200$$

$$3610200 \cdot s_3 = 63\mathbf{01}200$$

$$3610200 \cdot s_6 = 3610200$$

$$3610200 \cdot \pi_0 = \mathbf{0}610200$$

$$0610200 \cdot \pi_0 = 0610200$$

Is it possible to define an analogue of H_n^0 for the rook monoid ?

[Solomon 2004] Iwahori-Hecke ring of M_n gives a deformation of the rook monoid.

The 0-rook monoid as a transformation monoid

Bubble sort operators $\pi_1, \dots, \pi_{n-1}$:

$$(r_1 \dots r_n) \cdot \pi_i = \begin{cases} r_1 \dots r_{i-1} r_{i+1} r_i r_{i+2} \dots r_n & \text{if } r_i < r_{i+1}, \\ r_1 \dots r_n & \text{otherwise,} \end{cases}$$

Deletion operator π_0 : $(r_1 \dots r_n) \cdot \pi_0 = \quad 0 r_2 \dots r_n$.

Example:

$$3610200 \cdot \pi_1 = \mathbf{63}10200$$

$$6310200 \cdot \pi_1 = 6310200$$

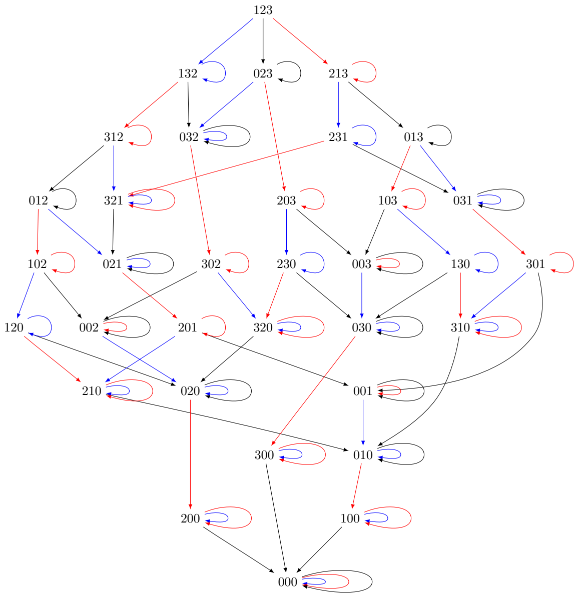
$$3601200 \cdot \pi_3 = 63\mathbf{10}200$$

$$3610200 \cdot \pi_3 = 6310200$$

$$3610200 \cdot \pi_6 = 3610200$$

$$3610200 \cdot \pi_0 = \mathbf{0}610200$$

$$0610200 \cdot \pi_0 = 0610200$$


123
132
023
213
312
032
231
013
012
321
203
103
031
102
021
302
230
003
130
301
120
002
201
320
030
310
210
020
001
300
010
200
100
000

Presentation of the 0-Rook Monoid

$$
\begin{aligned}
\pi_i^2 &= \pi_i & 1 \le i \le n-1, && \text{(Idm)}\\
\pi_i\pi_j &= \pi_j\pi_i & 1 \le i,j \le n-1 \quad |i-j| \ge 2, && \text{(Com)}\\
\pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} & 1 \le i \le n-2. && \text{(Br)}\\
\pi_0^2 &= \pi_0 & && \text{(Idm0)}\\
\pi_0\pi_i &= \pi_i\pi_0 & 2 \le i \le n-1. && \text{(Com0)}\\
\pi_0\pi_1\pi_0\pi_1 &= \pi_1\pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0 & && \text{(Br0)}
\end{aligned}
$$

$$
\begin{aligned}
\pi_i^2 &= \pi_i & 0 \le i \le n-1, && \text{(Idm)}\\
\pi_i\pi_j &= \pi_j\pi_i & 1 \le i,j \le n-1 \quad |i-j| \ge 2, && \text{(Com)}\\
\pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} & 1 \le i \le n-2. && \text{(Br)}\\
\pi_0\pi_i &= \pi_i\pi_0 & 2 \le i \le n-1. && \text{(Com0)}\\
\pi_0\pi_1\pi_0\pi_1 &= \pi_1\pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0 & && \text{(Br0)}
\end{aligned}
$$

Warning !

The maps

$$(r_1 \dots r_n) \cdot P_k \quad = \quad 0 \dots 0 r_{k+1} \dots r_n.$$

belongs to R_n and R_n^0.

But, though the map

$$(r_1 \dots r_n) \cdot K_2 \quad = \quad r_1 0 r_3 \dots r_n.$$

belongs to R_n, it doesn't belongs to R_n^0 !

Idea of the proof

$$\boxed{\begin{matrix} n \\ \vdots \\ i \end{matrix}} := \begin{cases} 1 & \text{if } i > n, \\ \pi_n \ldots \pi_i & \text{if } 0 \leq i \leq n, \\ \pi_n \ldots \pi_1 \pi_0 \pi_1 \ldots \pi_i & \text{if } i < 0, \end{cases}$$

Proposition

Given a rook r the shortest lexicographically minimal word for π_r has the form

$$\pi_r = \boxed{\begin{matrix} 0 \\ \vdots \\ c_1 \end{matrix}} \cdot \boxed{\begin{matrix} 1 \\ \vdots \\ c_2 \end{matrix}} \cdot \cdots \cdot \boxed{\begin{matrix} n-1 \\ \vdots \\ c_n \end{matrix}},$$

for some $c = (c_1, \ldots, c_n)$.

Canonical reduced expression

Example : 30240

Index the zeros by the missing letters in decreasing order : 30_5240_1

$$
\begin{array}{cr}
12345 & \mathbf{1}_5 \\
0_1 2345 & \cdot \pi_0 \\
20_1 345 & \cdot \pi_1 \\
320_1 45 & \cdot \pi_2\pi_1 \\
3240_1 5 & \cdot \pi_3 \\
30_5 240_1 & \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1
\end{array}
$$

Conclusion : $\mathbf{1}_5 \cdot [\pi_0 \cdot \pi_1 \cdot \pi_2\pi_1 \cdot \pi_3 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\pi_1] = 30240.$

$$
\pi_{30240} = \boxed{\begin{matrix}0\\ \vdots\\ 0\end{matrix}} \cdot \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}} \cdot \boxed{\begin{matrix}2\\ \vdots\\ 1\end{matrix}} \cdot \boxed{\begin{matrix}3\\ \vdots\\ 3\end{matrix}} \cdot \boxed{\begin{matrix}4\\ \vdots\\ -1\end{matrix}}
$$

Example : using coset R_5^0/R_4^0

$$
\begin{array}{c}
30145 \\
\downarrow \pi_4
\end{array}
$$

Matsumoto theorem for rook monoids

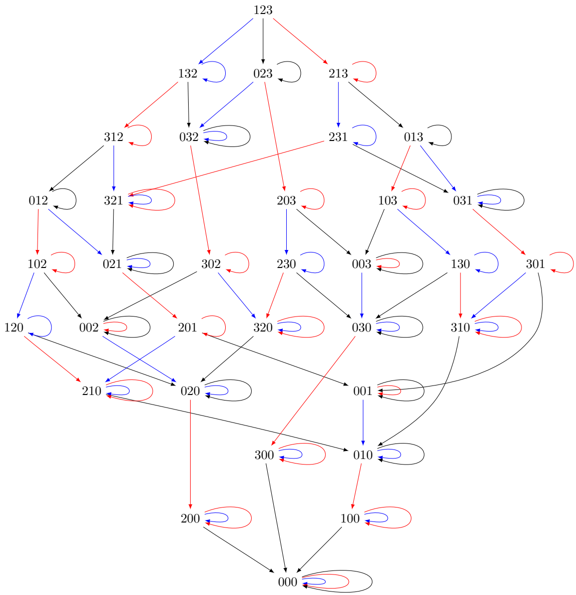
Theorem

Two reduced words give the same rook if and only if they can be related using only the relations:

$$
\begin{aligned}
s_i s_j &= s_j s_i & |i-j| &\geq 2, \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & 1 &\leq i \leq n-2. \\
\pi_0 s_i &= s_i \pi_0 & 2 &\leq i \leq n-1.
\end{aligned}
$$

Two reduced words give the same 0*-rook if and only if they can be related using only the relations:*

$$
\begin{aligned}
\pi_i \pi_j &= \pi_j \pi_i & |i-j| &\geq 2, \\
\pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} & 1 &\leq i \leq n-2.
\end{aligned}
$$


123
132
023
213
312
032
231
013
012
321
203
103
031
102
021
302
230
003
130
301
120
002
201
320
030
310
210
020
001
300
010
200
100
000

The right order of the 0-rook monoid

Theorem

The 0-rook monoid is $\mathcal{R}$-trivial and self-opposite therefore $\mathcal{J}$-trivial.

Theorem (Guilbaud-Rosenstiehl 1963)

The $\mathcal{R}$-order of the 0-rook monoid is a lattice.

The right order on permutations

$$\Delta := \{(b, a) \mid n \geq b > a > 0\}$$

$$\mathrm{Inv}(r) := \{(r_i, r_j) \mid i < j \text{ and } r_i > r_j > 0\} \subset \Delta.$$

Definition

$I \subseteq \Delta$ is transitive if $(c, b) \in I$ and $(b, a) \in I$ implies $(c, a) \in I$.

Lemma

$I = \mathrm{Inv}(\sigma)$ for some σ if and only if I and $\Delta \setminus I$ are both transitive. When this holds the permutation σ is unique.

Lemma

Let $\sigma, \tau \in \mathfrak{S}_n$, then $\sigma \leq_{\mathcal{R}} \tau$ if and only if $\mathrm{Inv}(\tau) \subseteq \mathrm{Inv}(\sigma)$.

Rook triple of a rook

Definition

Rook triple associated to r: $(\mathrm{supp}(r), \mathrm{Inv}(r), Z_r)$

- $\mathrm{supp}(r) :=$ *the set of non-zero letters appearing in r.*
- $\mathrm{Inv}(r) := \{(r_i, r_j) \mid i < j \text{ and } r_i > r_j > 0\}$
- $Z_r(\ell)$ *the number of* 0 *which appear after ℓ in r*

Example $r = 2054001$

- $\mathrm{supp}(r) = \{1, 2, 4, 5\}$;
- $\mathrm{Inv}(r) = \{(2, 1), (4, 1), (5, 4), (5, 1)\}$;
- $Z_r(1) = 0$, $Z_r(2) = 3$ and $Z_r(4) = Z_r(5) = 2$.

Characterization of rooks by their rook triple

A rook is characterized by its associated rook triple:

Proposition

A triple (S, I, Z) is the rook triple of a rook r if and only if:

- $I \subset \Delta \cap S^2$ *and* I *and* $(\Delta \cap S^2) \setminus I$ *are both transitive.*
- *for* $\ell \in S$, $0 \leq Z(\ell) \leq n - |S|$;
- *if* $(b, a) \in I$ *then* $Z(b) \geq Z(a)$ *else* $Z(b) \leq Z(a)$.

When this holds the rook r is unique.

The $\mathcal{R}$-order thanks to rook triples

Theorem

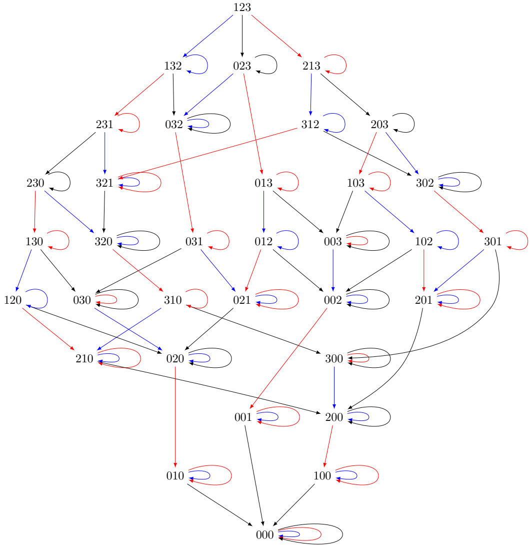
Let $r, u \in R_n$. Then $\pi_r \leq_{\mathcal{R}} \pi_u$ if and only if

- $\mathrm{supp}(r) \subseteq \mathrm{supp}(u)$
- $\{(b, a) \in \mathrm{Inv}(u) \mid b \in \mathrm{supp}(r)\} \subseteq \mathrm{Inv}(r)$
- $Z_u(\ell) \leq Z_r(\ell)$ *for* $\ell \in \mathrm{supp}(r)$.

In particular $\leq_{\mathcal{R}}$ is an order. R_n^0 is $\mathcal{R}$-trivial.

Note: meet and join thanks to rook triples.

The rook monoid in 3D (Note $\mathcal{L}$-graph)



Geometric point of view

Remark: Interpreting rook vectors as coordinates leads to a graph drawn on a polytope.

Definition

Stellohedron : convex hull of the rooks. Extremal points:

$$\mathrm{Stell}_n := \{\mathfrak{S}_n(0\dots0k\dots n) \mid k = 1,\dots,n\}$$

[Manneville-Pilaud] Graph associahedron of a star graph.

Is there an associated semigroup ?
It there lattice ?

The stellar lattice

The stellar monoid is $\mathcal{J}$-trivial (quotient of a $\mathcal{J}$-trivial).

Theorem

The $\mathcal{L}$-order of the stellar monoid is a **sub-lattice** *(i.e. stable by join and meet) of the $\mathcal{L}$-order of the 0-rook monoid.*

The stellar Monoid

Modify π_0 : kills the first letter and all the smaller letters:

$$361452 \cdot \pi_0 = 060450$$
$$460503 \cdot \pi_0 = 060500$$
$$060500 \cdot \pi_0 = 060500$$

Theorem

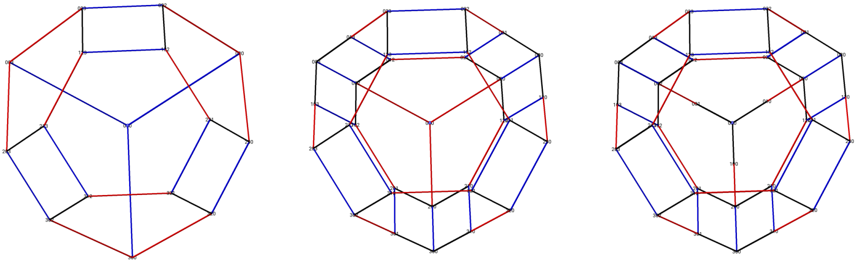
The stellar monoid S_n^0 is the quotient of the 0-rook monoid by the relations

$$\pi_i \pi_{i-1} \dots \pi_1 \pi_0 \pi_i \equiv \pi_i \pi_{i-1} \dots \pi_1 \pi_0$$

for $i < n - 1$.

Experiments thanks to James Mitchell's `libsemigroups`.

Some intermediate steps



Sequence of lattices inclusion

$$\{0^n\} = \mathrm{St}_0(R_n) \subset \mathrm{St}_1(R_n) \subset \mathrm{St}_2(R_n) \subset \cdots \subset \mathrm{St}_n(R_n) = R_n$$

Sequence of monoid quotients

$$\{0^n\} = \mathrm{St}_0(R_n^0) \leftarrow \mathrm{St}_1(R_n^0) \leftarrow \mathrm{St}_2(R_n^0) \leftarrow \cdots \leftarrow \mathrm{St}_n(R_n^0) = R_n^0$$

Thanks for your attention !