Generation of intermediate and partial map monoids of first-order structures

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Examples of infinitely generated semigroups

- Infinitely generated free semigroup/rectangular band, infinite left/right zero semigroups
- Classical transformation semigroups on an infinite set $\mathbb{N}$:
  - symmetric group $\mathrm{Sym}(\mathbb{N})$
  - full transformation monoid $\mathrm{End}(\mathbb{N})$
  - symmetric inverse monoid $\mathrm{Inv}(\mathbb{N})$
  - partial transformation monoid $\mathrm{Part}(\mathbb{N})$
  - injective (surjective) transformation monoid $\mathrm{Mon}(\mathbb{N})$ ($\mathrm{Epi}(\mathbb{N})$)
  - Baer-Levi semigroup $\mathcal{BL}(\mathbb{N})$ of injective transformations α where $|\mathbb{N} \setminus \mathbb{N}\alpha| = \aleph_0$
- Other examples exist (any uncountable semigroup)!

Cofinality, strong cofinality

Throughout this talk, S is an infinitely generated semigroup unless stated otherwise.

Definition (Cofinality)

The **cofinality** $\mathrm{cf}(S)$ of S is the least cardinal λ such that there exists a chain of proper subsemigroups $(U_i)_{i<\lambda}$ where $\bigcup_{i<\lambda} U_i = S$.

Definition (Strong cofinality)

The **strong cofinality** $\mathrm{scf}(S)$ of S is the least cardinal κ such that there exists a chain of proper sub*sets* $(V_i)_{i<\kappa}$ such that for all $i < \kappa$ there exists a $j < \kappa$ such that $V_i V_i \subseteq V_j$ and $S = \bigcup_{i<\kappa} V_i$.

It is true that $\mathrm{cf}(S) \geq \mathrm{scf}(S) \geq \aleph_0$.

The Bergman property for semigroups

Definition (Cayley boundedness, Bergman property)

Say that S is **semigroup Cayley bounded** with respect to a set U that generates S as a semigroup if $S = U \cup U^2 \cup \ldots \cup U^n$ for some $n \in \mathbb{N}$.
S has the **semigroup Bergman property** (BP) if it is Cayley bounded for *every* generating set U of S.

Drop the 'semigroup' from now on!

Theorem 1 (Maltcev, Mitchell, Ruškuc '09)

(1) $\mathrm{scf}(S) > \aleph_0$ if and only if S has the BP and $\mathrm{cf}(S) > \aleph_0$.

(2) If $\mathrm{scf}(S) > \aleph_0$, then $\mathrm{scf}(S) = \mathrm{cf}(S)$.

Examples

(1) $\mathrm{cf}(S) = \mathrm{scf}(S) > \aleph_0$, BP

- $\mathrm{Sym}(\mathbb{N})$
- $\mathrm{End}(\mathbb{N})$, $\mathrm{Inv}(\mathbb{N})$, $\mathrm{Part}(\mathbb{N})$
- $\mathrm{Aut}(R)$, $\mathrm{End}(R)$

(3) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$, BP

- Infinitely generated rectangular band
- Infinite left zero semigroup
- $S = \langle X \mid xyz = xy \rangle$ with X infinite

(2) $\mathrm{cf}(S) > \mathrm{scf}(S) = \aleph_0$, $\neg$BP

- bounded symmetric group of the rationals $\mathrm{BSym}(\mathbb{Q})$

(4) $\mathrm{cf}(S) = \mathrm{scf}(S) = \aleph_0$, $\neg$BP

- Free semigroup X^* with X infinite
- Baer-Levi semigroup $\mathcal{BL}(\mathbb{N})$

Cofinality toolbox

Lemma 2 (TC+, 2017)

(1) If S is countable, then $\mathrm{cf}(S) = \aleph_0$.

(2) Let T be an infinitely generated subsemigroup of S and I an ideal of S such that $S = T \sqcup I$. Then $\mathrm{cf}(S) \leq \mathrm{cf}(T)$.

Definition (Relative rank)

Suppose that S is any semigroup and A is a subset of S. The **relative rank** $\mathrm{rank}(S : A)$ **of** S **modulo** A is the minimum cardinality of a set B such that $\langle A \cup B \rangle = S$.

Lemma 3 (2·Pech, 13)

Let T be an infinitely generated subsemigroup of S. If $\mathrm{cf}(T) > \aleph_0$ and $\mathrm{rank}(S : T)$ is finite then $\mathrm{cf}(S) > \aleph_0$.

Monomorphisms and epimorphisms

Look at $\mathrm{Mon}(\mathbb{N})$ and $\mathrm{Epi}(\mathbb{N})$.

Theorem 4 (Mitchell, Péresse 11)

$\mathrm{rank}(\mathrm{Mon}(\mathbb{N}) : \mathrm{Sym}(\mathbb{N})) = 2$ and $\mathrm{rank}(\mathrm{Epi}(\mathbb{N}) : \mathrm{Sym}(\mathbb{N})) = 5$. Both of these semigroups do not have the Bergman property.

As $\mathrm{cf}(\mathrm{Sym}(\mathbb{N})) > \aleph_0$, we can conclude from Lemma 3 and Theorem 1:

Consequence

$\mathrm{cf}(\mathrm{Mon}(\mathbb{N})) > \mathrm{scf}(\mathrm{Mon}(\mathbb{N})) = \aleph_0$. Same holds for $\mathrm{Epi}(\mathbb{N})$.

So both $\mathrm{Mon}(\mathbb{N})$ and $\mathrm{Epi}(\mathbb{N})$ live in case (2).

Strong cofinality toolbox

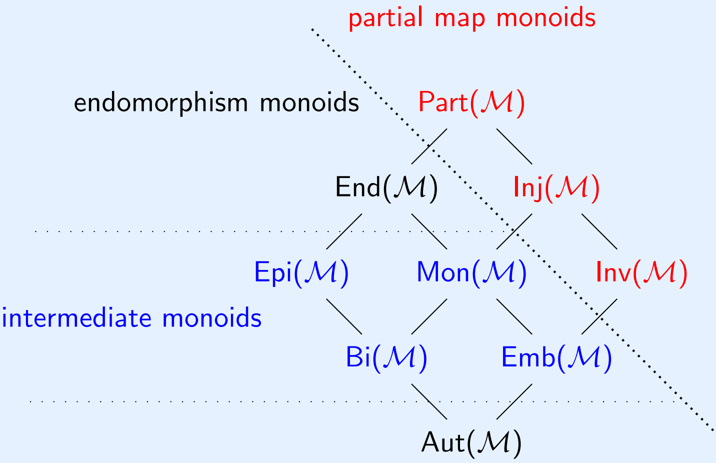
Proposition 5 (TC+, 2017)

Suppose that S has an infinite descending chain of ideals $S = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$ and assume that $J = \bigcap_{i \in \mathbb{N}} I_i$ is non-empty. Let $L_i = I_i \setminus I_{i+1}$ and suppose also that $L_i L_j \subseteq (\bigcup_{n=0}^{h} L_n) \cup J$ for some $h \in \mathbb{N}$. Then $\mathrm{scf}(S) = \aleph_0$.

A trip to the shop is required!

Intermediate and partial map monoids

For $\mathcal{M}$, a countably infinite relational first-order structure:



Discrete linear order 1/2

Let $(\mathbb{N}, \leq)$ be the discrete linear order; $\mathrm{Emb}(\mathbb{N}, \leq)$ is the monoid of injective order-preserving transformations of $\mathbb{N}$. Here, $\mathrm{Aut}(\mathbb{N}, \leq)$ is the trivial group.

Fun fact!

For any coinfinite subset A of $\mathbb{N}$, there exists a unique embedding α such that $\mathbb{N} \setminus \mathbb{N}\alpha = A$.

For $k \in \mathbb{N}$, let α_k be the unique map such that $\mathbb{N} \setminus \mathbb{N}\alpha_k = \{k\}$.

Consequences of the fun fact

- $|\mathrm{Emb}(\mathbb{N}, \leq)| = 2^{\aleph_0}$.
- Any generating set for $\mathrm{Emb}(\mathbb{N}, \leq)$ contains α_k for all $k \in \mathbb{N}$.

Discrete linear order 2/2

$$F := \{\beta \in \text{Emb}(\mathbb{N}, \leq) \ : \ |\mathbb{N} \setminus \mathbb{N}\beta| < \aleph_0\}$$
$$J_\infty := \{\gamma \in \text{Emb}(\mathbb{N}, \leq) \ : \ |\mathbb{N} \setminus \mathbb{N}\gamma| = \aleph_0\}$$

Then $\text{Emb}(\mathbb{N}, \leq) = F \sqcup J_\infty$, and the countable submonoid F is infinitely generated. So

Proposition 6 (TC+, 2017)

(1) $\text{cf}(\text{Emb}(\mathbb{N}, \leq)) = \text{scf}(\text{Emb}(\mathbb{N}, \leq)) = \aleph_0$.

(2) The generating set $X = \{\alpha_k \ : \ k \in \mathbb{N}\} \cup J_\infty \cup \{e\}$ of $\text{Emb}(\mathbb{N}, \leq)$ is not Cayley bounded.

So $\text{Emb}(\mathbb{N}, \leq)$ falls in case (4).

Random graph 1/3

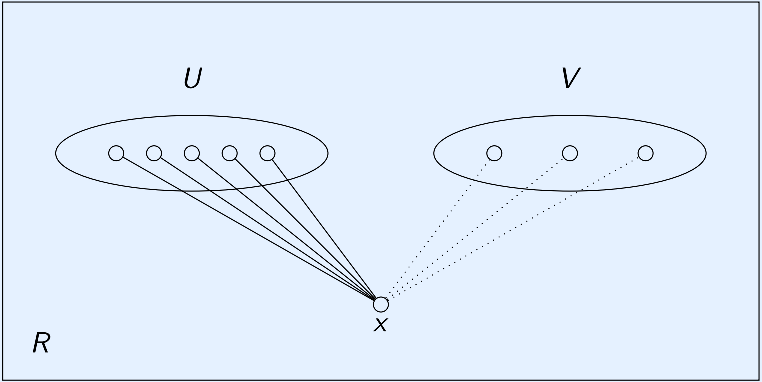


Figure : The random graph R

R is nice. Look at $\mathrm{Bi}(R)$, $\mathrm{Emb}(R)$, and $\mathrm{Mon}(R)$.

Random graph 2/3

Ideals are important!

- Let $I_k \subseteq \mathrm{Bi}(R)$ (or $\mathrm{Mon}(R)$) be the ideal of all bimorphisms (monomorphisms) that add in $\geq k$ edges for $k \in \mathbb{N} \cup \{\infty\}$.
- Let $J_k \subseteq \mathrm{Emb}(R)$ be the ideal of all embeddings that omit $\geq k$ vertices for $k \in \mathbb{N} \cup \{\infty\}$.

These form infinite descending chains of ideals that match the conditions of Proposition 4. So:

Proposition 7 (TC+)

Let R be the random graph and $T \in \{\mathrm{Bi}(R), \mathrm{Emb}(R), \mathrm{Mon}(R)\}$. Then $\mathrm{scf}(T) = \aleph_0$.

Random graph 3/3

$\text{Bi}(R) \smallsetminus I_\infty$ is generated by some bimorphism α that adds in a single edge together with $\text{Aut}(R)$. Consequently:

Proposition 8 (TC+, 2017)

(1) $\text{rank}(\text{Bi}(R) \smallsetminus I_\infty : \text{Aut}(R)) = 1$, and so $\text{cf}(\text{Bi}(R) \smallsetminus I_\infty) > \aleph_0$.

(2) The generating set $X = \text{Aut}(R) \cup \{\alpha\} \cup I_\infty$ of $\text{Bi}(R)$ is not Cayley bounded.

Similarly, $\text{cf}(\text{Emb}(R) \smallsetminus J_\infty) > \aleph_0$, and both $\text{Emb}(R)$ and $\text{Mon}(R)$ do not have the Bergman property. Also, $\text{cf}(\text{FMon}(R)) > \aleph_0$, where $\text{FMon}(R)$ is the monoid of monomorphisms of R that leave out finitely many (possibly zero) edges and vertices.

Question

Are $\text{cf}(\text{Bi}(R))$, $\text{cf}(\text{Emb}(R))$ and $\text{cf}(\text{Mon}(R))$ all uncountable?

A reminder

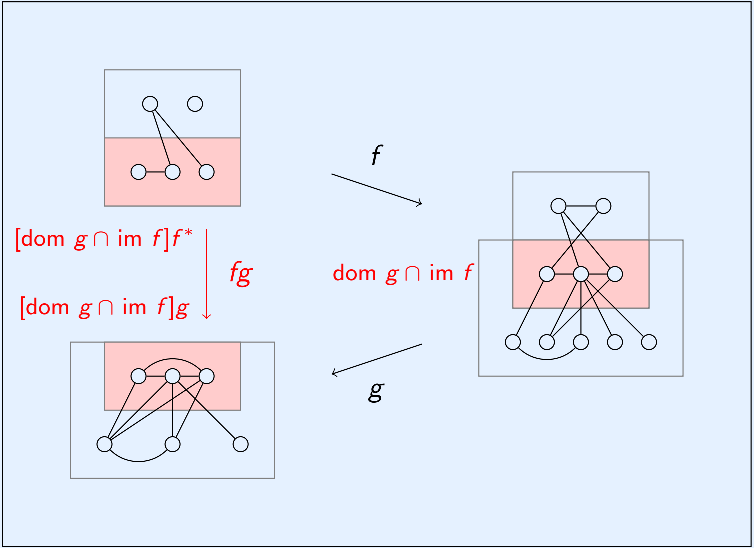
For a countably infinite relational first-order structure $\mathcal{M}$:

- $\mathsf{Inv}(\mathcal{M})$ is the symmetric inverse monoid of $\mathcal{M}$; the monoid of all isomorphisms between substructures of $\mathcal{M}$.
- $\mathsf{Part}(\mathcal{M})$ is the partial homomorphism monoid of $\mathcal{M}$.
- $\mathsf{Inj}(\mathcal{M})$ is the partial monomorphism monoid of $\mathcal{M}$.

Fun aside!

Much like $\mathsf{Bi}(\mathcal{M}) \subseteq \mathsf{Sym}(M)$ is a group-embeddable monoid that isn't a group, $\mathsf{Inj}(\mathcal{M}) \subseteq \mathsf{Inv}(M)$ is a inverse semigroup-embeddable monoid that isn't an inverse semigroup.

Composition in Part(R)



A trip to the hardware shop

Definition (Strong distortion)

A semigroup S is **strongly distorted** if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of natural numbers and $N_S \in \mathbb{N}$ such that for all sequences $(s_n)_{n\in\mathbb{N}}$ of elements from S there exist $t_1, t_2, \ldots, t_{N_S} \in S$ such that each s_n can be written as a product of length at most a_n in the elements $t_1, t_2, \ldots, t_{N_S}$.

Element of $(s_n)_{n\in\mathbb{N}}$	s_1	s_2	s_3	$\ldots$	s_n	$\ldots$
Length of product of t_i's equal to s_n	a_1	a_2	a_3	$\ldots$	a_n	$\ldots$

Figure : Strong distortion

Theorem 9 (MMR 09)

If S is strongly distorted, then $\mathrm{scf}(S) > \aleph_0$.

Sierpiński rank

Definition (Sierpiński rank)

The **Sierpiński rank** (SR) of S is defined to be the smallest natural number n (if it exists; ∞ otherwise) such that any countable sequence $(s_n)_{n\in\mathbb{N}}$ of elements in S is contained in an n-generated subsemigroup of S.

Examples

1 $(\mathbb{N}, +)$ has SR 1.

2 $\mathrm{Inv}(\mathbb{N})$ has SR 2.

3 Semigroup of increasing functions $f : [0,1] \to [0,1]$ has SR 3.

4+ $\mathrm{Mon}(\aleph_n)$ has SR $n+4$ for all $n \in \mathbb{N}_0$.

∞ $\mathcal{BL}(\mathbb{N})$ has infinite SR.

Every strongly distorted semigroup has finite SR; but converse is not true.

Big theorem

Theorem 10 (TC+, 2017)

Let $\mathcal{M}$ be a countable first-order structure such that:

(a) $\mathcal{M}$ contains substructures $\mathcal{M}_i$ (where $i \in \mathbb{N}_0$) with $\mathcal{M}_i \cong \mathcal{M}$, and it also contains substructures $\mathcal{N}_k = \bigsqcup_{i \geq k} \mathcal{M}_i$;

(b) there exists an isomorphism from $\mathcal{N}_0$ to $\mathcal{N}_1$ mapping each $\mathcal{M}_i$ to $\mathcal{M}_{i+1}$, and;

(c) for any countable sequence $(\hat{f}_i)_{i \in \omega}$ where each $\hat{f}_i$ is a partial isomorphism of $\mathcal{M}_i$, the union $\bigcup_{i \in \omega} \hat{f}_i : \bigcup_{i \in \omega} \operatorname{dom} \hat{f}_i \to \bigcup_{i \in \omega} \operatorname{im} \hat{f}_i$ is a partial isomorphism of $\mathcal{M}$.

Then $\operatorname{scf}(\operatorname{Inv}(\mathcal{M})) > \aleph_0$. Similarly, $\operatorname{scf}(\operatorname{Inj}(\mathcal{M}))$, $\operatorname{scf}(\operatorname{Part}(\mathcal{M})) > \aleph_0$.

If conditions (a)–(c) hold, the SR of $\operatorname{Inv}(\mathcal{M})$ is at most 3. Similarly, the SR of $\operatorname{Inj}(\mathcal{M})$ and $\operatorname{Part}(\mathcal{M})$ are at most 5.

Examples and a non-example

Examples

- $(\mathbb{Q}, <)$ satisfies conditions (a)–(c), and so $\mathrm{Inv}(\mathbb{Q}, <) = \mathrm{Inj}(\mathbb{Q}, <) = \mathrm{Part}(\mathbb{Q}, <)$ has a SR of 3 and an uncountable strong cofinality.
- The generic digraph D without 2-cycles satisfies conditions (a)–(c): here, $\mathrm{Inv}(D)$ and $\mathrm{Inj}(D) = \mathrm{Part}(D)$ have uncountable strong cofinality.
- The random graph R and the generic poset $\mathbb{P}$ satisfies conditions (a)–(c) for all types of finite partial map.

Non-example

- $(\mathbb{N}, \leq)$ does not satisfy condition (c).

Semilattice of idempotents

For an inverse semigroup S there is a semilattice of idempotents $E(S)$. For $\text{Inv}(\mathcal{M})$, the idempotents $E = E(\text{Inv}(\mathcal{M}))$ are identity maps on substructures.

If $\mathcal{M}$ is infinite, then $|E| = 2^{\aleph_0}$ and is an infinitely generated semigroup; so you can subject it to the same analysis.

Like $(\mathbb{N}, \leq)$, there is a unique element of E for every subset of $\mathbb{N}$.

Proposition 11 (TC+, 2017)

$\text{cf}(E) = \text{scf}(E) = \aleph_0$, and E does not have the Bergman property.

Questions (?)

- Work on cofinality of $\mathrm{Bi}(R)$ and others; are they uncountable? Do they have finite SR?
- What overgroup G of $\mathrm{Aut}(R) \leq G \leq \mathrm{Sym}(VR)$ is generated by $\mathrm{Bi}(R)$ and 'inverses'?
- Investigate semigroup theory of $\mathrm{Inj}(M)$ and $\mathrm{Part}(\mathcal{M})$. Is there a structural analogue for the binary relation monoid?
- If S is uncountable, is there a connection between uncountable cofinality and finite SR?