Semigroup of Tropical Matrices

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On a semigroup S, the relation $\leq_{\mathcal{R}^*}$ is defined by the rule that for $a, b \in S$, $a \leq_{\mathcal{R}^*} b$ if and only if

$$xb = yb \Rightarrow xa = ya \text{ and } xb = b \Rightarrow xa = a$$

for all $x, y \in S$. Clearly $\leq_{\mathcal{R}^*}$ is a quasi-order on S.
The equivalence relation associated with $\leq_{\mathcal{R}^*}$ is denoted by $\mathcal{R}^*$ and it is a left congruence on S.
It follows, $a \,\mathcal{R}\, b$ implies that $a \,\mathcal{R}^*\, b$ and, $e \,\mathcal{R}\, f$ if and only if $e \,\mathcal{R}^*\, f$ for all e, $f \in E(S)$.
The relation $\mathcal{L}^*$ is defined dually and the relations $\mathcal{H}^*$ and $\mathcal{D}^*$ by putting

$$\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*; \; \mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*.$$

For any $a \in S$, the set of left identities for a in $E \subseteq E(S)$ is denoted by ${}_E a$, that is,

$$ {}_E a = \{e \in E : ea = a\}. $$

The relation $\leq_{\widetilde{\mathcal{R}}}$ on S is defined by the rule that for $a, b \in S$, $a \leq_{\widetilde{\mathcal{R}}} b$ if and only if for all $e \in E$

$$ eb = b \Rightarrow ea = a. $$

The equivalence relation associated with $\leq_{\widetilde{\mathcal{R}}}$ is denoted by $\widetilde{\mathcal{R}}_E$. We have following result:

Lemma

For any $a \in S$ and $e \in E$, $a \, \widetilde{\mathcal{R}} \, e$ if and only if $ea = a$ and for all $f \in E$, if $fa = a$ then $fe = e$.

Extension of Green's Relations

We say that S is

- left E-abundant if every $\mathcal{R}^*$-class contains an idempotent of E;
- Weakly left E-abundant if every $\widetilde{\mathcal{R}}$-class contains an idempotent of E;
- *abundant* if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class of S contains an idempotent;
- A semigroup S is *weakly abundant* (Fountain Semigroup!) if each $\widetilde{\mathcal{R}}$-class and each $\widetilde{\mathcal{L}}$-class of S contains an idempotent.

For basic facts about the relations and abundant and weakly abundant semigroups we refer to:

1. J. Fountain, A class of right PP monoids, Quart. J. Math. Oxford (2) 28 (1977), pp 285-300.
2. J. Fountain, Adequate semigroups, Proc. Edinb. Math. Soc. (2) 22 (1979), pp 113-125.

Semirings

A *hemiring* [resp. *semiring*] is a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

1. $(S, +)$ is a commutative monoid with identity element 0;
2. $(S, \cdot)$ is a semigroup [resp. monoid with identity element 1_S];
3. Multiplication distributes over addition from either side;
4. $0r = 0 = r0$ for all $r \in S$.

As a rule, we will write 1 instead of 1_S when there is no likelihood of confusion.
Note that if $1 = 0$ then $r = r1 = r0 = 0$ for each element r of S and so $S = \{0\}$. In order to avoid this trivial case, we will assume that all semirings under consideration are nontrivial, i.e.

5. $1 \neq 0$.

Information Algebras

A semiring S is said to be *zerosumfree* if and only if $r + r' = 0$ implies that $r = r' = 0$.

Every additively idempotent semiring is zerosum free.

If S is a zerosumfree semiring then

$$S' = \{0\} \cup \{r \in S : rb \neq 0 \text{ for all } 0 \neq b \in S\}$$

is a subsemiring of S.

A nonzero element a of a semiring S is a *left(right) zero divisor*if and only if there exists a nonzero element b of S satisfying $ab = 0(ba = 0)$.

It is a *zero divisor* if and only if it is a left and a right zero divisor.

A semiring S having no zero divisors is *entire*.

If S is an entire, zerosumfree semiring then$S^* = S - \{0\}$ is a subsemiring of S.

Entire zerosumfree semirings are called *information algebras*.

Semigroup of Matrices over Semirings

Theorem

A division semiring S is either zerosumfree or is a division ring.

Let $M_n(S)$ denote the set of all $n \times n$ matrices over a semiring S and the induced operation of multiplication be defined as:

$$(AB)_{ij} = \sum_{k=1}^{n} (A_{ik} B_{kj})$$

for all $A, B \in M_n(S)$.

Theorem

Let S be a semiring and $n \in \mathbb{N}$. Then:

1. *$M_n(S)$ is a semigroup under induced operation of multiplication;*
2. *Set of upper triangular matrices where entries below the diagonal are zero, denoted $U_n(S)$ is a subsemigroup of $M_n(S)$.*

Semigroup of Matrices over Semirings

Let Γ be a reflexive, transitive relation on $[n] := \{1, ..., n\}$, and define

$$\Gamma(S) = \{A \in M_n(S) : A_{ij} \neq 0 \Leftrightarrow (i,j) \in \Gamma\}.$$

Theorem

Let S be an information algebra and $M_n(S)$ be the set of $n \times n$ matrices over S. Then the set $\Gamma(S)$ is a subsemigroup of $M_n(S)$.

Definition

A semiring S is said to be *exact* (or *FP-injective*) if for all $A \in S^{m \times n}$:

(E1) If $\mathbf{x} \in S^{1 \times n} \setminus Row(A)$, then there exist $\mathbf{u},\mathbf{v} \in S^{n \times 1}$ such that $A\mathbf{u} = A\mathbf{v}$, but $\mathbf{xu} \neq \mathbf{xv}$.

(E2) If $\mathbf{x} \in S^{m \times 1} \setminus Col(A)$, then there exist $\mathbf{u},\mathbf{v} \in S^{1 \times m}$ such that $\mathbf{u}A = \mathbf{v}A$, but $\mathbf{ux} \neq \mathbf{vx}$.

Example of Exact Semirings

The class of exact semirings is known to contain:

- all fields;
- all proper quotients of principal ideal domains;
- all matrix rings and finite group rings over the above;
- the Boolean semiring $\mathbb{B}$, the tropical semiring $\mathbb{T}$, and some generalisations of these.

For more information on semirings and exact semirings we refer to:

1. J. S. Golan, Semirings and their Applications, Springer Science Business Media, 1999.
2. D. Wilding, M. Johnson, M. Kambites, Exact rings and semirings, J. Algebra 388 (2013), 324-337.

Two matrices $A, B \in M_n(S)$ are $\mathcal{L}$-related, written $A \mathcal{L} B$, if there are $M, P \in M_n(S)$ with $MA = B$ and $PB = A$. OR $A \mathcal{L} B$ if and only if $Row(A) = Row(B)$. Similarly $A, B \in M_n(S)$ are $\mathcal{R}$-related, written $A \mathcal{R} B$, if there are $N, Q \in M_n(S)$ with $AN = B$ and $BQ = A$. Again, we notice that $A \mathcal{R} B$ if and only if $Col(A) = Col(B)$.

Theorem

Let S be an exact semiring. Then $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{L} = \mathcal{L}^*$ in the semigroup $M_n(S)$. Thus $M_n(S)$ is abundant if and only if it is regular.

Tropical Semifield

We focus here on $S = \mathbb{FT}$. The finitary tropical (or max-plus) semifield $\mathbb{FT}$ has elements from $\mathbb{R}$ with binary operations defined as:

$$\begin{aligned} x \oplus y &= max(x, y); \text{ and} \\ x \otimes y &= x + y. \end{aligned}$$

We see that $(\mathbb{FT}, \oplus, \otimes)$ is an idempotent semifield.
Its generalisations are $(\mathbb{T}, \oplus, \otimes)$ and $(\overline{\mathbb{T}}, \oplus, \otimes)$, where $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and $\overline{\mathbb{T}} = \mathbb{R} \cup \{-\infty, \infty\}$.

Conventions

Here we have some conventions:

$$\begin{aligned} a \oplus -\infty &= a = -\infty \oplus a, \\ a \oplus \infty &= \infty = \infty \oplus a, \\ \infty \oplus -\infty &= \infty = -\infty \oplus \infty, \\ a \otimes -\infty &= -\infty = -\infty \otimes a, \\ a \otimes \infty &= \infty = \infty \otimes a, \\ \infty \otimes -\infty &= -\infty = -\infty \otimes \infty, \end{aligned}$$

for all $a \in \mathbb{FT}$.

Let $M_n(\mathbb{S})$ denotes the set of all $n \times n$ matrices over $\mathbb{S} \in \{\mathbb{FT}, \mathbb{T}, \overline{\mathbb{T}}, \}$, with multiplication $\otimes$ defined as:

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} (A_{ik} \otimes B_{kj})$$

for all $A, B \in M_n(\mathbb{S})$.
Then $(M_n(\mathbb{S}), \otimes)$ is a semigroup.

Tropical Affine n-Space

Let $\mathbb{S}^n$ denote the set of all real n-tuples $\vec{v} = (v_1, ..., v_n)$ with obvious operations of addition and scalar multiplication:

$$(\vec{v} \oplus \vec{w})_i = v_i \oplus w_i,$$

$$(\lambda \otimes \vec{v})_i = \lambda \otimes v_i.$$

Semigroup of Tropical Matrices

For $A \in M_n(\mathbb{S})$

- The *row space* $R(A) \subseteq \mathbb{S}^n$ is defined as the tropical submodule of $\mathbb{S}^n$ generated by the rows of A.

And similarly,

- The column space $C(A) \subseteq \mathbb{S}^n$ is defined as the tropical submodule $\mathbb{S}^n$ generated by the columns of A.

Example

Let

$$A = \begin{bmatrix} 2 & 4 & 5 \\ 5 & 3 & 7 \\ 9 & \infty & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 6 & 7 \\ -\infty & 4 & 9 \\ 6 & 3 & 3 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 11 & 8 & 13 \\ 13 & 11 & 12 \\ 16 & \infty & \infty \end{bmatrix}.$$

Theorem

Let $A, B \in M_n(\mathbb{S})$ *for* $\mathbb{S} \in \{\mathbb{FT}, \mathbb{T}, \bar{\mathbb{T}}\}$. *Then*

1. $A \preceq_{\mathcal{L}} B \Leftrightarrow R_{\mathbb{S}}(A) \subseteq R_{\mathbb{S}}(B)$;
2. $A\mathcal{L}B \Leftrightarrow R_{\mathbb{S}}(A) = R_{\mathbb{S}}(B)$;
3. $A \preceq_{\mathcal{R}} B \Leftrightarrow C_{\mathbb{S}}(A) \subseteq C_{\mathbb{S}}(B)$;
4. $A\mathcal{R}B \Leftrightarrow C_{\mathbb{S}}(A) = C_{\mathbb{S}}(B)$;
5. $A\mathcal{H}B \Leftrightarrow C_{\mathbb{S}}(A) = C_{\mathbb{S}}(B)$ *and* $R_{\mathbb{S}}(A) = R_{\mathbb{S}}(B)$;
6. $A\mathcal{D}B$ *if and only if* $C_{\mathbb{S}}(A)$ *and* $C_{\mathbb{S}}(B)$ *are isomorphic as* $\mathbb{S}$*-modules;*
7. $A\mathcal{D}B$ *if and only if* $R_{\mathbb{S}}(A)$ *and* $R_{\mathbb{S}}(B)$ *are isomorphic as* $\mathbb{S}$*-modules.*

C. Hollings and M. Kambites, Tropical matrix duality and Green's D relation, Journal of the London Mathematical Society, 86 (2012), pp. 520-538.

Upper Triangular Tropical Matrices

Let $U_n(\mathbb{T})$ be the subset of all *upper triangular tropical matrices* in $M_n(\mathbb{T})$, where $M_{ij} = -\infty$ for all $i > j$ and $M \in U_n(\mathbb{T})$.

- This set $U_n(\mathbb{T})$ is a subsemigroup of $M_n(\mathbb{T})$ under its operation of multiplication;
- The set of matrices in $U_n(\mathbb{T})$ where all entries on and above the diagonal are finite forms a subsemigroup of $U_n(\mathbb{T})$, which we denote by $U_n(\mathbb{FT})$;
- Many structural properties of upper triangular tropical matrix semigroups have been worked out with details in Taylor's thesis and some articles are also available on semigroup identities which hold in triangular case

References

1. Z. Izhakian, Semigroup identities in the monoid of triangular tropical matrices, Semigroup Forum, 88 (2014), no. 1, 145-161.
2. J. Okninski, Identities of the semigroup of upper triangular tropical matrices. Communications in Algebra, 43 (2015), pp. 4422-4426.
3. M. Taylor, On upper triangular tropical matrix semigroups, tropical matrix identities and T-modules, Thesis submitted to the University of Manchester (2016).
4. Z. Izhakian, Erratum to: Semigroup identities in the monoid of triangular tropical matrices, Semigroup Forum, 92 (2016), p733.
5. L. Daviaud, M. Johnson, M. Kambites, Identities in Upper Triangular Tropical Matrix Semigroups and the Bicyclic Monoid, preprint, (2016).

Known Results

In his thesis, Taylor has shown following results:

- Every $M \in U_n(\mathbb{FT})$ has both row and column rank n
- For $M, N \in U_n(\mathbb{FT})$, $M\mathcal{R}N$ (respectively $M\mathcal{L}N$) if and only if the ith row (column) of N is a scaling of the ith row (column) of M.
- Green's relations for $U_n(\mathbb{FT})$ are the restrictions of corresponding relations on $M_n(\mathbb{T})$
- $U_2(\mathbb{FT})$ has only one $\mathcal{D}$-class, in fact, $U_2(\mathbb{FT})$ is an inverse semigroup
- $U_n(\mathbb{FT})$ has only one $\mathcal{J}$-class, for all n.

Comments and Examples

$U_2(\mathbb{T})$ is not an inverse semigroup.
Recall that a semigroup S is inverse if and only if it is regular with commuting idempotents.

Example

Let

$$E = \begin{bmatrix} 0 & -\infty \\ 0 & -\infty \end{bmatrix}, \; F = \begin{bmatrix} -\infty & -\infty \\ 0 & 0 \end{bmatrix}.$$

Then E, F are idempotents in $U_n(\mathbb{T})$ and

$$EF = \begin{bmatrix} -\infty & -\infty \\ -\infty & -\infty \end{bmatrix} \neq \begin{bmatrix} -\infty & -\infty \\ 0 & -\infty \end{bmatrix} = FE.$$

Comments and Examples

1. $U_n(\mathbb{FT})$ is not regular for $3 \leq n$.

Example

Let $M \in U_3(\mathbb{FT})$

$$M = \begin{bmatrix} 0 & 1 & 0 \\ -\infty & 0 & 2 \\ -\infty & -\infty & 0 \end{bmatrix}$$

If there is some $N \in U_3(\mathbb{FT})$ such that $MNM = M$, then we must have $N_{11} = N_{22} = N_{33} = 0$ and

$$\begin{aligned} \max\{1, N_{12}\} &= 1; \\ \max\{2, N_{23}\} &= 2\ ; \\ \max\{N_{13}, (1 + N_{23}), 3, (2 + N_{12})\} &= 0, \end{aligned}$$

which is not possible for any choice of N. Thus M is not regular.

Idempotents in $U_n(\mathbb{FT})$

By definition, a matrix E is an idempotent in $U_n(\mathbb{FT})$ exactly if

$$E_{ij} = \bigoplus_{k=1}^{n} (E_{ik} \otimes E_{kj})$$

But this says that:

$$\begin{aligned} E_{ii} + E_{ii} &= E_{ii} \text{ or } E_{ii} = 0 \text{ for each } 1 \leq i \leq n \quad \text{and} \\ E_{ik} \otimes E_{kj} &\leq E_{ij} \text{ for each } k,\ 1 \leq i < k < j \leq n. \end{aligned}$$

Idempotents in $U_n(\mathbb{FT})$

Theorem

For $E, F \in \mathbf{E}(U_n(\mathbb{FT}))$, $(EF)^{[\frac{n+1}{2}]} = (FE)^{[\frac{n+1}{2}]}$.

Theorem

Let $A \in U_n(\mathbb{FT})$. Then for any $X \in U_n(\mathbb{FT})$, $XA = A$ exactly if

$$X_{ij} \leq \min_{j \leq k \leq n} (A_{ik} - A_{jk}),\ 1 \leq i < j \leq n;\ X_{ii} = 0,\ 1 \leq i \leq n.$$

Idempotents in $U_n(\mathbb{FT})$

Theorem

For every $A \in U_n(\mathbb{FT})$ there exists a unique idempotent matrix in $U_n(\mathbb{FT})$ denoted ${}_AE$ such that ${}_AEA = A$ and if $FA = A$ then $FE = E$.

This shows that every $A \in U_n(\mathbb{FT})$ is $\widetilde{\mathcal{R}}$-related to a unique idempotent and hence $U_n(\mathbb{FT})$ is a Fountain semigroup!

- For $A \in U_n(\mathbb{FT})$, ${}_{\bar{E}}\mathbf{A}$ is a subsemigroup of $\bar{\mathbf{E}}(U_n(\mathbb{FT}))$.

Theorem

Let

$$\bar{\mathbf{E}}(U_n(\mathbb{FT})) = \{X \in U_n(\mathbb{FT}) : X_{ii} = 0,\ 1 \leq i \leq n\}.$$

Then $\bar{\mathbf{E}}(U_n(\mathbb{FT}))$ is a subsemigroup and $\bar{\mathbf{E}}(U_n(\mathbb{FT})) = (\mathbf{E}(U_n(\mathbb{FT})))^{(n-1)}$.

Abundance in $U_n(\mathbb{FT})$

Theorem

Let $A, B \in U_n(\mathbb{FT})$*. Then* $A \, \mathcal{R} \, B$ *exactly if* $A \, \mathcal{R}^* \, B$*.*

Theorem

Let $A, B \in U_n(\mathbb{FT})$*. Then* $CA \, \tilde{\mathcal{R}} \, CB$ *for all* $C \in U_n(\mathbb{FT})$ *exactly if* $A \, \mathcal{R} \, B$*.*

Questions

1. For S exact, what conditions on S and n are necessary/sufficient for $M_n(S)$ to be regular?
2. Can we characterise $\mathcal{R}^*$ and $\mathcal{L}^*$ in $M_n(S)$ for some non-exact examples S?
3. Are there non-exact semirings S for which $M_n(S)$ is abundant if and only if it is regular? Or does the exactness condition completely characterise these cases?
4. What about upper triangular matrix semigroups over S exact? (Or non-exact!)


Thank you