Universal words and sequences

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January 18, 2017

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Universal words

Let S be a semigroup, let A an alphabet, and let $w \in A^+$. Then w is a semigroup universal word for S if for every element $s \in S$, there is a homomorphism $\phi : A^+ \to S$ such that $(w)\phi = s$.

Let G be a group, let A be an alphabet, and let $w \in F(A)$. Then w is a group universal word for G if for every element $g \in G$, there is a homomorphism $\phi : F(A) \to G$ such that $(w)\phi = g$.

Examples

Let $A = \{a, b\}$ be an alphabet.

Example

Let S be a semigroup. Then a is a semigroup universal word for S.

Example

Let M be a monoid. Then ab is a semigroup universal word for M.

Example

Let S be a band. Then any word $w \in A^+$ is a semigroup universal word for S.

Non-example

Let X be a set. Denote the symmetric group on X by $\mathrm{Sym}(X)$.

Question: Is a^2 a group universal word for $\mathrm{Sym}(X)$?

Some known results

Theorem (Mycielski '87, Lyndon '90)
Let A be an alphabet, and let X be an infinite set. Then a word $w \in F(A)$ is a group universal word for $\mathrm{Sym}(X)$ *if and only if $w \neq u^n$ for any non-empty $u \in F(A)$ and $n > 1$.*

Conjecture (Droste and Truss '06)
Let A be an alphabet, and let R be the Rado graph. Then a word $w \in F(A)$ is a group universal word for $\mathrm{Aut}(R)$ *if and only if $w \neq u^n$ for any non-empty $u \in F(A)$ and $n > 1$.*

Let X be an infinite set, and let X^X denote the transformation monoid on X.

Question Let A be an alphabet, and let $w \in A^+$. When is w a semigroup universal word for the transformation monoid on the set X?

Transformation monoid

Proposition (Isbell '66, Hyde and J '14)

Let X be a countable set, let A be an alphabet, and let $w \in A^+$ be such that no proper prefix of w is also a suffix of w. Then w a semigroup universal word for X^X.

Example

The word $a^n b^n$ for any $n \in \mathbb{N}$ is semigroup universal word for X^X.

S_w

Let A be an alphabet. For $w \in A^+$, let S_w be the smallest submonoid of A^* such that

- if $w = svuvq$ for some $s, q \in S_w$ and $v, u \in A^*$, then $v \in S_w$;
- if $w = svt$ for some $s, v, t \in A^*$ such that $sv, vt \in S_w$, then $w \in S_w$.

Proposition

S_w exists, and there is an algorithm to determine S_w.

Transformation monoid

Theorem (Hyde and J '14)

Let X be a countable set, and let $w \in A^+$. Let p and s be the longest words in S_w such that $w = pus$ for some $u \in A^*$. Suppose that u is not a subword of p and $w \notin S_w$. Then w is a semigroup universal word for X^X.

Example

If no proper prefix of w is a suffix of w, then w is universal.

Example

Let $w = aba^2b^2ab$. Then w is a semigroup universal word for X^X.

Revisiting an old result

Theorem (Mycielski '87, Lyndon '90)
Let A be an alphabet, and let X be an infinite set. Then a word $w \in F(A)$ is a group universal word for $\mathrm{Sym}(X)$ *if and only if $w \neq u^n$ for any non-empty $u \in F(A)$ and $n > 1$.*

Open question. Given $w \in A^*$ is it decidable whether w is a semigroup universal word for X^X?

Open question. Does the set of all semigroup universal words for X^X change with the cardinality of X?

Universal sequences

Universal sequence

Theorem (Sierpiński '35)

Let X be a countable set. For all $f_1, f_2, \ldots \in X^X$, there are $a, b \in X^X$ such that $f_n = a^2b^3(abab^3)^{n+1}ab^2ab^3$

Remark

Other expressions for f_n are known, for example James Hyde shown that $f_n = a(ab)^n b$ for all $n \in \mathbb{N}$.

Let S be a semigroup, let A be an alphabet, and let I be a countable set. Then $\{w_i : i \in I\} \subseteq A^+$ is a semigroup universal sequence for S, if for every sequence $\{s_i : i \in I\} \subseteq S$, there is a homomorphism $\phi : A^+ \to S$ such that $(w_i)\phi = s_i$ for all $i \in I$.

A semigroup universal list of length n is defined as above, except I is of size n.

Theorem (Hyde, J '14)

Let $\{w_n : n \in \mathbb{N}\} \subseteq \{a, b\}^+$ such that w_n is a subword of w_m only if $n = m$. Suppose that for every $n, m \in \mathbb{N}$, a prefix p of w_n is a suffix of w_m only if $p = w_n = w_m$. Then $\{w_n : n \in \mathbb{N}\}$ is a semigroup universal sequence for X^X.

Theorem (Hyde, J, Péresse, Mitchell '15)

$\mathrm{Aut}(\mathbb{Q}, \leq)$ *has a semigroup universal sequence over* 2 *letter alphabet.*

Theorem (Hyde, J, Péresse, Mitchell)

Let X be countable. If S is either symmetric inverse monoid over X, dual symmetric inverse monoid over X, or partition monoid over X, then S has a semigroup universal sequence over 2 *letter alphabet.*

Open questions:

- Given $n > 2$, is there an uncountable semigroup which has a universal sequence over alphabet of size n, but not $n - 1$?
- Does automorphism group of the Rado graph have a finite universal sequence? What about other Fraïssé limits?

Proposition (Hyde and J '16)
The automorphism group of the Rado graph has a semigroup universal list of any finite length over 4 *letters.*