A Brief History of Semigroup Representations

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Outline

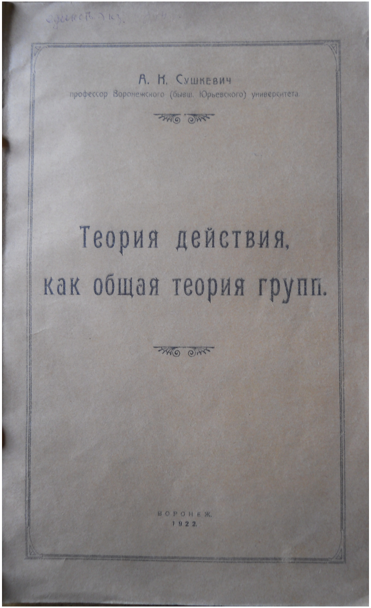
1. Early trends in semigroup theory
2. Sushkevich
3. Clifford
4. Munn
5. Ponizovskii
6. Parallel developments and changing viewpoints

Semigroup theory to 1950ish

- Sushkevich and ‘generalised substitutions’
- Factorisation in semigroups by analogy with rings
- Completely simple semigroups
- Dubreil and ‘demi-groupes’
- Inverse semigroups

A. K. Sushkevich (1889–1961)

Sushkevich and generalised groups



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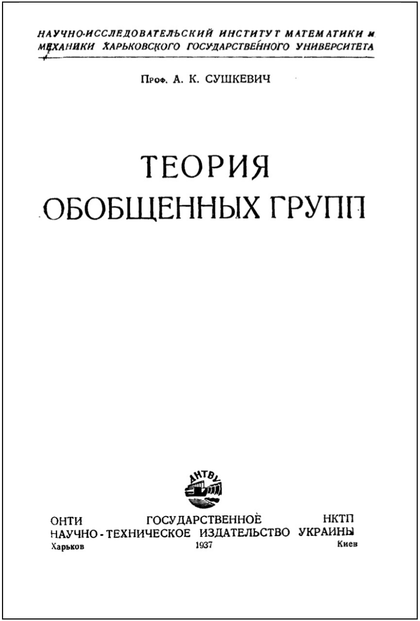
Теория действия, как общая теория групп.

ВОРОНЕЖ. 1922.



НАУЧНО-ИССЛЕДОВАТЕЛЬСКИЙ ИНСТИТУТ МАТЕМАТИКИ и МЕХАНИКИ ХАРЬКОВСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА

Проф. А. К. СУШКЕВИЧ

ТЕОРИЯ ОБОБЩЕННЫХ ГРУПП

ОНТИ ГОСУДАРСТВЕННОЕ НКТП

НАУЧНО-ТЕХНИЧЕСКОЕ ИЗДАТЕЛЬСТВО УКРАИНЫ

Харьков 1937 Киев

Sushkevich and generalised groups

Über die Darstellung der eindeutig nicht umkehrbaren Gruppen mittelst der verallgemeinerten Substitutionen.

Von Prof. **A. Suschkewitsch** (Woronesch).

Es handelt sich um endliche Gruppen, deren Operation eindeutig, associativ, aber nicht eindeutig umkehrbar ist. Ebenso, wie man die gewöhnlichen abstrakten Gruppen konkret in der Form der Substitutionsgruppen darstellt, und diese Darstellung bei der Untersuchung dieser Gruppen gewisse Vorteile bietet, — ist es auch vorteilhaft auch für unsere verallgemeinerten Gruppen eine solche konkrete Darstellung zu haben. Dazu eignen sich sehr die sogenannten verallgemeinerten Substitutionen der n Symbole, d. h. solche Substitutionen, bei denen verschiedene Symbole in ein und dasselbe Symbol übergehen können, die, also, in der gewöhnlichen Form dargestellt, in der unteren Zeile nicht alle Symbole zu haben brauchen, ein und dasselbe Symbol dagegen mehr als ein Mal haben können. Solche Substitutionen können nach derselben Regel wie die gewöhnlichen mit einander komponiert werden, und es ist leicht zu sehen, dass bei dieser Komposition wohl das associative Gesetz, dagegen nicht das Gesetz der eindeutigen Umkehrbarkeit gilt. Da überdies die Anzahl aller solchen Substitutionen der n Symbole endlich ist ($=n^n$), so können aus ihnen endliche Gruppen der von uns betrachteten Art gebildet werden. Wir werden aber zeigen, dass diese Substitutionsgruppen auch alle Arten unserer verallgemeinerten Gruppen erschöpfen, oder, anders ausgedrükt:

Jede abstrakte Gruppe der von uns betrachteten Art kann als Gruppe der verallgemeinerten Substitutionen dargestellt werden.

D. h. man kann eine Gruppe der verallgemeinerten Substitutionen konstruieren, die einstufig isomorph der gegebenen abstrakten Gruppe ist.

Bekanntlich, kann man zu einer gewöhnlichen abstrakten Gruppe folgendermassen die ihr einstufig isomorphe Substitutionsgruppe konstruieren: es bestehe die gegebene abstrakte Gruppe n^{ter} Ordnung aus den Elementen $A_1, A_2, \ldots A_n$; dann soll dem Elemente $A_k (k = 1, 2, \ldots n)$ die Substitution:

$$\overline{A}_k = \begin{pmatrix} A_1 & A_2 & \ldots A_n \\ A_1 A_k & A_2 A_k & \ldots A_n A_k \end{pmatrix} \qquad (1)$$

entsprechen. Die n Substitutionen $\overline{A}_1, \overline{A}_2, \ldots \overline{A}_n$ bilden eine Gruppe, die der gegebenen Gruppe einstufig isomorph ist.

Schlagen wir auch bei unserer verallgemeinerten Gruppe denselben Weg ein, so bekommen wir auch hier zu jedem Elemente eine ihm entsprechende Substitution der

Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit.

Von

Anton Suschkewitsch in Woronesch (Rußland).

Einleitung.

In der vorliegenden Abhandlung habe ich den Versuch gemacht eine abstrakte Theorie der endlichen Gruppen, deren Operation nicht eindeutig umkehrbar ist, zu konstruieren. Freilich sind in der mathematischen Literatur solche Gruppen in konkreter Form schon betrachtet worden. Als Beispiel solcher konkreten Gruppen kann man die Theorie der nichtkommutativen Ringe, speziell auch die Theorie der hyperkomplexen Zahlen anführen, wobei auch das Analogon zu der Aufspaltung des besonderen Teiles, den ich als „Kern“ bezeichne, durchgeführt ist[1]). Dabei werden aber zugleich zwei Operationen betrachtet: die „Addition“ und die „Multiplikation“. Es entsteht nun die Frage nach der Verallgemeinerung, die man erhält, wenn man die eine Operation — nämlich die Addition — wegläßt und bloß die andere — die Multiplikation — beibehält, die als eindeutig, assoziativ, aber nicht eindeutig umkehrbar vorausgesetzt wird.

Die Darstellung, die ich im folgenden einführe, ist von diesen konkreten Fällen völlig unabhängig. Ich bleibe fortwährend im Gebiete der reinen Gruppentheorie, betrachte also nur eine einzige Operation in einer völlig abstrakten Form und beschränke meine Betrachtungen ausschließlich auf *endliche* Gruppen mit einer eindeutigen assoziativen Operation.

Bekanntlich hat das Gesetz der eindeutigen Umkehrbarkeit zwei Seiten: das „linke“ Gesetz: „Aus der Gleichung $BA = CA$ folgt $B = C$“; das „rechte“ Gesetz: „Aus $AB = AC$ folgt $B = C$“. Sofern über die Kommutativität der Operation keine Voraussetzung gemacht wird, sind diese beiden Seiten völlig unabhängig voneinander.

[1]) Vgl. Maclagan-Wedderburn, On hypercomplex numbers, Proceedings of the London Math. Soc. (2) 6 (1908). Ich bin auf diese Arbeit erst nach Fertigstellung der meinigen durch einen freundlichen Hinweis von Frl. E. Noether aufmerksam geworden.

Sushkevich and generalised groups

'Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit' (1928):

Considered a finite right cancellative semigroup $\mathfrak{A}$ is a finite right cancellative semigroup.

Showed that for any P in $\mathfrak{A}$, $\mathfrak{A}P = \mathfrak{A}$, but $P\mathfrak{A} \subsetneq \mathfrak{A}$, in general.

Named a finite right cancellative semigroup a left group.

In a left group $\mathfrak{A}$, every idempotent E is a right identity.

Let $E_1, E_2, \dots, E_s$ be all the right identities of $\mathfrak{A}$. Then

$$\mathfrak{A} = \bigcup_{\kappa=1}^{s} E_\kappa \mathfrak{A},$$

where the $\mathfrak{C}_\kappa := E_\kappa \mathfrak{A}$ are disjoint isomorphic groups. Moreover, the collection of all right identities of $\mathfrak{A}$ forms a semigroup, the left principal group $\mathfrak{E} = \{E_1, E_2, \dots, E_s\}$ under the multiplication $E_\kappa E_\lambda = E_\kappa$.

Sushkevich and generalised groups

Let $\mathfrak{G}$ be an arbitrary finite semigroup.

Consider the subsets $\mathfrak{G}P$, as P runs through all elements of $\mathfrak{G}$; choose subset $\mathfrak{G}X$ of smallest size, denote this by $\mathfrak{A}$.

$\mathfrak{A}$ is clearly a minimal left ideal of $\mathfrak{G}$ — and a left group.

All minimal left ideals $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_r$ of $\mathfrak{G}$ are isomorphic to $\mathfrak{A}$.

By structure of left groups:

$$\mathfrak{A}_\kappa = \mathfrak{C}_{\kappa 1} \cup \mathfrak{C}_{\kappa 2} \cup \cdots \cup \mathfrak{C}_{\kappa s},$$

where the $\mathfrak{C}_{\kappa\lambda}$ are disjoint isomorphic groups.

Similarly for minimal right ideals $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_s$:

$$\mathfrak{B}_\lambda = \mathfrak{C}_{1\lambda} \cup \mathfrak{C}_{2\lambda} \cup \cdots \cup \mathfrak{C}_{r\lambda}.$$

Furthermore $\mathfrak{C}_{\kappa\lambda} = \mathfrak{A}_\kappa \cap \mathfrak{B}_\lambda$.

Sushkevich and generalised groups

Define kernel of $\mathfrak{G}$:

$$\mathfrak{K} = \bigcup_{\kappa=1}^{r} \mathfrak{A}_\kappa = \bigcup_{\lambda=1}^{s} \mathfrak{B}_\lambda = \bigcup_{\kappa=1}^{r} \bigcup_{\lambda=1}^{s} \mathfrak{C}_{\kappa\lambda}.$$

$$\begin{array}{ccc|ccccccc}
\mathfrak{K} & = & & \mathfrak{A}_1 & \cup & \mathfrak{A}_2 & \cup & \cdots & \cup & \mathfrak{A}_r \\
\| & & & \| & & \| & & & & \| \\ \cline{4-10}
\mathfrak{B}_1 & = & & \mathfrak{C}_{11} & \cup & \mathfrak{C}_{21} & \cup & \cdots & \cup & \mathfrak{C}_{r1} \\
\cup & & & \cup & & \cup & & & & \cup \\
\mathfrak{B}_2 & = & & \mathfrak{C}_{12} & \cup & \mathfrak{C}_{22} & \cup & \cdots & \cup & \mathfrak{C}_{r2} \\
\cup & & & \cup & & \cup & & & & \cup \\
\vdots & & & \vdots & & \vdots & & \vdots & & \vdots \\
\cup & & & \cup & & \cup & & & & \cup \\
\mathfrak{B}_s & = & & \mathfrak{C}_{1s} & \cup & \mathfrak{C}_{2s} & \cup & \cdots & \cup & \mathfrak{C}_{rs}
\end{array}$$

Sushkevich and generalised groups

Thus, every finite semigroup $\mathfrak{G}$ contains a minimal ideal $\mathfrak{K}$, completely determined by

1. the structure of the abstract group $\mathfrak{C}$ that is isomorphic to the $\mathfrak{C}_{\kappa\lambda}$;
2. the numbers r and s;
3. the $(r-1)(s-1)$ products $E_{11}E_{\kappa\lambda}$ ($\kappa = 2, \ldots, r$, $\lambda = 2, \ldots, s$), where $E_{\kappa\lambda}$ denotes the identity of $\mathfrak{C}_{\kappa\lambda}$.

Can also choose 1–3 arbitrarily in order to construct a 'stand-alone' kernel, i.e., a finite simple semigroup.

Sushkevich and matrices

Über die Matrizendarstellung der verallgemeinerten Gruppen

ANTON SUSCHKEWITSCH, *Charkow*

Im Folgenden betrachte ich Matrizen, deren Rang kleiner als Ordnung ist; für die Komposition (Multiplikation) solcher Matrizen gilt bekanntlich das assoziative Gesetz, im Allgemeinen aber nicht das Gesetz der eindeutigen Umkehrbarkeit. Diese Matrizen eignen sich also zur Darstellung der Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit[1]). Über die Darstellung einiger Typen solcher endlicher Gruppen wird nun im Folgenden die Rede sein. Dabei führe ich im § 1. einige Hilfssätze über die Matrizen ein; im § 2 betrachte ich die Darstellung der gewöhnlichen (klassischen) Gruppen; im § 3 werden die Gruppen für deren Komposition nur die eine Seite des Gesetzes der eindeutigen Umkehrbarkeit gilt (die sogen. Links- oder Rechts-Gruppen), dargestellt; im § 4 wird die Darstellung der sogen. Kerngruppe[1]) betrachtet; zum Schluss kommt noch ein Beispiel zur Illustration der Theorie.

§ 1

Es wird uns häufig nötig sein die rechteckigen Matrizen, d. h. solche Matrizen, wo die Anzahl der Spalten nicht gleich der Anzahl der Zeilen ist, zu betrachten. Hat eine solche Matrix m Zeilen und n Spalten, so nennen wir sie eine „mn-Matrix". Eine mn-Matrix kann mit einer pq-Matrix dann und nur dann komponiert werden, wenn $n=p$ ist; das Produkt ist eine mq-Matrix. Ist speziell auch $m=q$, so ist dieses Produkt eine quadratische Matrix m^{ter} Ordnung.

Seien nun

$$A=\begin{pmatrix} a_{11}\dots a_{1n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{m1}\dots a_{mn}\end{pmatrix},\quad B=\begin{pmatrix} b_{11}\dots b_{1q} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n1}\dots b_{nq}\end{pmatrix}$$

zwei rechteckige Matrizen vom Range n; $n\leq m$, $n\leq q$.

Satz 1. AB ist vom Range n.

Beweis. Sind, z. B.:

$$A=\begin{vmatrix} a_{\alpha_1 1}\,a_{\alpha_1 2}\dots a_{\alpha_1 n} \\ a_{\alpha_2 1}\,a_{\alpha_2 2}\dots a_{\alpha_2 n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{\alpha_n 1}\,a_{\alpha_n 2}\dots a_{\alpha_n n}\end{vmatrix}\neq 0,\qquad B=\begin{vmatrix} b_{1\beta_1}\,b_{1\beta_2}\dots b_{1\beta_n} \\ b_{2\beta_1}\,b_{2\beta_2}\dots b_{2\beta_n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ b_{n\beta_1}\,b_{n\beta_2}\dots b_{n\beta_n}\end{vmatrix}\neq 0,$$

so ist auch $AB\neq 0$; das ist, aber, eine Determinante n^{ter} Ordnung von AB. Dagegen sind alle Determinanten, $(n+1)^{ter}$ Ordnung von AB gleich Null (das folgt, z. B., aus dem verallgemeinerten Multiplikationssatze der Determinanten).

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1) Über diese Gruppen, speziell über die Kerngruppe. s. meine Arbeit: „Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit". Math. Ann. Bd. 99.

Sushkevich and matrices

'Über die Matrizendarstellung der verallgemeinerten Gruppen' (1933):

Theorem: All representations of an ordinary (finite) group by means of $m \times m$ matrices of rank $n < m$ may be obtained from the representations of the same group by $n \times n$ matrices of rank n.

Theorem: All representations of a left group $\mathfrak{G}$ by means of $m \times m$ matrices of rank $n < m$ may be obtained from the representations of the group $\mathfrak{A}_\kappa$ by $n \times n$ matrices of rank n.

Characterisation of matrix representations of finite simple semigroups follow.

Sushkevich and matrices

'Über eine Verallgemeinerung der Semigruppen' (1935):

Take set $\mathfrak{X}$ with binary operation.

Suppose that $\mathfrak{X}$ has a subset $\mathfrak{G}$ that forms a cancellative semigroup.

Distinguish two different types of elements of $\mathfrak{X}$, K-elements and L-elements, such that

1. each K-element is composable on the left, with well-defined result, with any element of $\mathfrak{G}$;
2. no K-element is composable on the right with any element of $\mathfrak{G}$;
3. if $X \in \mathfrak{X}$ is composable on the left, but not on the right, with an element of $\mathfrak{G}$, then X is a K-element;
4. each L-element is composable on the right, with well-defined result, with any element of $\mathfrak{G}$;
5. no L-element is composable on the left with any element of $\mathfrak{G}$;
6. if $Y \in \mathfrak{X}$ is composable on the right, but not on the left, with an element of $\mathfrak{G}$, then Y is an L-element;
7. K-elements are not composable with each other;
8. L-elements are not composable with each other;
9. a K-element and an L-element are composable with each other, in either order, with well-defined result in each case.

Begins to make sense if you think about matrices...

Sushkevich and matrices

'On groups of matrices of rank 1' (1937):

For field P, take vectors $(a_1, \ldots, a_n) \in P^n$ such that $a_1^2 + \cdots + a_n^2 = 1$.

Form elements $A = (a, a')\alpha$, where (a, a') is an ordered pair of such vectors, and α is a scalar factor from P.

The collection of all elements $A = (a, a')\alpha$, together with 0, denoted by $\mathfrak{H}$.

Two non-zero elements $A = (a, a')\alpha$ and $B = (b, b')\beta$ deemed equal precisely when $a = b$, $a' = b'$ and $\alpha = \beta$.

Compose (non-zero) elements A, B according to the rule $AB = (a, b')\alpha\beta(a' \cdot b)$, where $a' \cdot b$ denotes the scalar product of a' and b.

Sushkevich and matrices

'On groups of matrices of rank 1' (1937):

Sushkevich studied different collections of elements associated with a vector pair (a, a'): $\mathfrak{G}_{a,a'}$ (forming an ordinary group) and $\mathfrak{N}_{a,a'}$ (a zero semigroup).

Put

$$\mathfrak{A}_a = \left(\bigcup_{\substack{x \\ x\cdot a \neq 0}} \mathfrak{G}_{x,a} \right) \cup \left(\bigcup_{\substack{y \\ y\cdot a = 0}} \mathfrak{N}_{y,a} \right), \; \mathfrak{B}_a = \left(\bigcup_{\substack{x \\ x\cdot a \neq 0}} \mathfrak{G}_{a,x} \right) \cup \left(\bigcup_{\substack{y \\ y\cdot a = 0}} \mathfrak{N}_{a,y} \right).$$

So $\mathfrak{A}'_a = \bigcup_{\substack{x \\ x\cdot a\neq 0}} \mathfrak{G}_{x,a}$, is a left group.

Then

$$\mathfrak{H} = \{0\} \cup \left(\bigcup_b \mathfrak{A}_b \right) = \{0\} \cup \left(\bigcup_a \mathfrak{B}_a \right),$$

a generalised group of kernel type (a.k.a. a completely 0-simple semigroup).

A. H. Clifford (1908–1992)

Clifford and matrix representations

MATRIX REPRESENTATIONS OF COMPLETELY SIMPLE SEMIGROUPS.*

By A. H. Clifford.

By a semigroup is meant a system S of elements $a, b, \cdots$ closed under a single binary associative operation:

$$(ab)c = a(bc).$$

To each element a of S let there correspond a uniquely determined matrix $T(a)$ with n rows and columns and with elements in a (commutative) field Ω. If, for all a, b in S,

$$T(ab) = T(a)T(b),$$

then the correspondence $\mathfrak{T}: a \to T(a)$ is called a (matrix) representation of S in Ω of degree n. The notions of equivalence, reduction, and decomposition are defined exactly as in the theory of representations of groups or algebras.

The only work dealing with representations of semigroups, of which the author is aware, is that of Suschkewitsch. In [2] he makes considerable progress in the determination of all representations of a type of finite semigroup which he calls a Kerngruppe. [3] gives an abstraction of the process used in [2], at the same time removing the finiteness restriction. In a previous paper [1], he determined the structure of all possible Kerngruppen.

The latter determination has recently been extended and simplified by Rees [4]. In Rees's terminology, a Kerngruppe is a completely simple (finite) semigroup without a zero element. He characterizes every completely simple semigroup as a regular matrix semigroup over a group with zero (definitions in § 2). In the present paper we show how to construct all representations of a (not necessarily regular) matrix semigroup over a group with zero; the process is summarized at the close of § 6. In a final section (§ 8) we show how a Brandt groupoid [5] can be made into a matrix semigroup of especially simple type by the adjunction of a zero element; the theory of the present paper is then applied to find all its representations.

1. **Factorizations of a matrix of finite rank.** This first section is concerned with a problem in pure matrix theory, namely that of finding all solutions X, Y of the matric equation

$$XY = H,$$

* Received March 18, 1941.

BASIC REPRESENTATIONS OF COMPLETELY SIMPLE SEMIGROUPS.*[1]

By A. H. Clifford.

1. **Introduction.** In a previous paper [1], the author discussed the theory of representations of a completely simple semigroup S by matrices over a field Ω. According to the fundamental theorem of Rees [2], S is isomorphic with, and hence may be taken to be, a regular matrix semigroup over a group with zero. It was shown in [1] that every representation $\mathfrak{T}^*$ of S induces a representation $\mathfrak{T}$ of G; we call $\mathfrak{T}^*$ an extension of $\mathfrak{T}$ to S.

A given representation $\mathfrak{T}$ of G may not be extendible to a representation $\mathfrak{T}^*$ of S; but if it is so extendible, then the extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ of least possible degree over Ω is uniquely determined by $\mathfrak{T}$ to within equivalence. We call $\mathfrak{T}_0^*$ the basic extension of $\mathfrak{T}$, and by a basic representation of S we shall mean one that is the basic extension to S of a representation of G. Any extension $\mathfrak{T}^*$ of a representation $\mathfrak{T}$ of G reduces (but does not in general decompose) into the basic extension $\mathfrak{T}_0^*$ of $\mathfrak{T}$ and null representations.

It is immediate from Theorems 4.1 and 6.1 of [1] that the mapping $\mathfrak{T} \to \mathfrak{T}_0^*$ is one-to-one (in the sense of equivalence) from the extendible representations of G to the basic representations of S. However, several questions concerning this correspondence were left unanswered. It was shown (Theorem 7.1) that if $\mathfrak{T}$ is irreducible, so is $\mathfrak{T}_0^*$, but the converse was left open. One of the main purposes of this note is to prove that the converse is true, and hence that *all the irreducible representations of S over Ω are obtained as the basic extensions to S of the extendible irreducible representations of G.*

In § 2 we show that the correspondence $\mathfrak{T} \to \mathfrak{T}_0^*$ preserves decomposition. In § 3 we show that it preserves reduction in a limited sense: the *non-null* irreducible constituents of $\mathfrak{T}_0^*$ are the basic extensions of the irreducible constituents of $\mathfrak{T}$. An example in § 4 shows that an extraneous null constituent can occur in $\mathfrak{T}_0^*$. (Thanks to W. D. Munn for pointing this out.)

* Received July 18, 1959.

[1] This paper was prepared with the partial support of the National Science Foundation grant to the Tulane Mathematics Department.

Clifford and matrix representations

'Matrix representations of completely simple semigroups' (1942):

A (matrix) representation of a semigroup S is a morphism $\mathcal{I} : S \to M_n(\Omega)$, where $M_n(\Omega)$ denotes the multiplicative semigroup of $n \times n$ matrices with entries from a field Ω; $T(a)$ denotes the matrix to which $a \in S$ corresponds.

Clifford and matrix representations

'Matrix representations of completely simple semigroups' (1942):

Take completely 0-simple semigroup S, represented as Rees matrix semigroup with elements written in form $(a)_{i\lambda}$.

Normalise sandwich matrix P in such a way that all entries are either 0 or e; in particular, arrange so that $p_{11} = e$.

Then $(a)_{11}(b)_{11} = (ab)_{11}$, hence $\{(a)_{11}\}$ forms a 0-group $G_1 \cong G^0$.

Clifford and matrix representations

'Matrix representations of completely simple semigroups' (1942):

Any matrix representation $\mathcal{I}^* : (a)_{i\lambda} \mapsto T^*[(a)_{i\lambda}]$ of a completely 0-simple semigroup S induces a representation of G_1, which may be transformed in such a way that

$$T^*[(a)_{11}] = \begin{pmatrix} T(a) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathcal{I} : a \mapsto T(a)$ is a proper representation of G^0:

$$T(a)T(b) = T(ab), \qquad T(e) = I, \qquad T(0) = 0,$$

for all $a, b \in G$; $\mathcal{I}^*$ is an extension of $\mathcal{I}$ from G to S. Also:

$$T^*[(e)_{i1}] = \begin{pmatrix} T(p_{1i}) & 0 \\ R_i & 0 \end{pmatrix} \quad \text{and} \quad T^*[(e)_{1\lambda}] = \begin{pmatrix} T(p_{\lambda 1}) & Q_\lambda \\ 0 & 0 \end{pmatrix},$$

for suitable matrices R_i and Q_λ, for which it may be shown that $R_1 = Q_1 = 0$. Put $H_{\lambda i} = T(p_{\lambda i}) - T(p_{\lambda 1} p_{1i})$.

Clifford and matrix representations

'Matrix representations of completely simple semigroups' (1942):

Theorem: Let $\mathcal{I}$ be a proper representation of G^0. Then

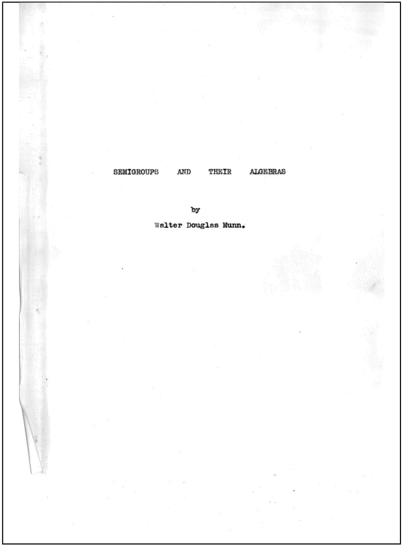
$$T^* \left[(a)_{i\lambda}\right] = \begin{pmatrix} T(p_{1i}ap_{\lambda 1}) & T(p_{1i}a)Q_\lambda \\ R_i T(ap_{\lambda 1}) & R_i T(a)Q_\lambda \end{pmatrix}$$

defines a representation $\mathcal{I}^*$ of S if and only if $Q_\lambda R_i = H_{\lambda i}$, for all i, λ. Conversely, every representation of S is equivalent to one of this form.

Provides procedure for construction of all representations of a completely 0-simple semigroup from those of its structure group.

W. Douglas Munn (1929–2008)

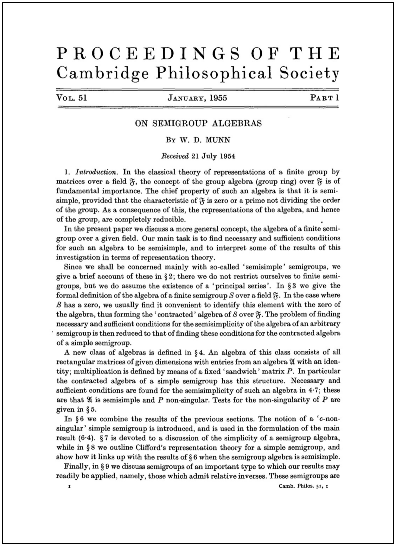
Munn and semigroup algebras



SEMIGROUPS AND THEIR ALGEBRAS

by

Walter Douglas Munn.



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ON SEMIGROUP ALGEBRAS

BY W. D. MUNN

Received 21 July 1954

1. *Introduction.* In the classical theory of representations of a finite group by matrices over a field $\mathfrak{F}$, the concept of the group algebra (group ring) over $\mathfrak{F}$ is of fundamental importance. The chief property of such an algebra is that it is semi-simple, provided that the characteristic of $\mathfrak{F}$ is zero or a prime not dividing the order of the group. As a consequence of this, the representations of the algebra, and hence of the group, are completely reducible.

In the present paper we discuss a more general concept, the algebra of a finite semi-group over a given field. Our main task is to find necessary and sufficient conditions for such an algebra to be semisimple, and to interpret some of the results of this investigation in terms of representation theory.

Since we shall be concerned mainly with so-called 'semisimple' semigroups, we give a brief account of these in §2; there we do not restrict ourselves to finite semi-groups, but we do assume the existence of a 'principal series'. In §3 we give the formal definition of the algebra of a finite semigroup S over a field $\mathfrak{F}$. In the case where S has a zero, we usually find it convenient to identify this element with the zero of the algebra, thus forming the 'contracted' algebra of S over $\mathfrak{F}$. The problem of finding necessary and sufficient conditions for the semisimplicity of the algebra of an arbitrary semigroup is then reduced to that of finding these conditions for the contracted algebra of a simple semigroup.

A new class of algebras is defined in §4. An algebra of this class consists of all rectangular matrices of given dimensions with entries from an algebra $\mathfrak{A}$ with an iden-tity; multiplication is defined by means of a fixed 'sandwich' matrix P. In particular the contracted algebra of a simple semigroup has this structure. Necessary and sufficient conditions are found for the semisimplicity of such an algebra in 4·7; these are that $\mathfrak{A}$ is semisimple and P non-singular. Tests for the non-singularity of P are given in §5.

In §6 we combine the results of the previous sections. The notion of a 'c-non-singular' simple semigroup is introduced, and is used in the formulation of the main result (6·4). §7 is devoted to a discussion of the simplicity of a semigroup algebra, while in §8 we outline Clifford's representation theory for a simple semigroup, and show how it links up with the results of §6 when the semigroup algebra is semisimple.

Finally, in §9 we discuss semigroups of an important type to which our results may readily be applied, namely, those which admit relative inverses. These semigroups are

Munn and semigroup algebras

“In the theory of representations of a finite group G by matrices over a field $\mathfrak{F}$ the concept of the algebra of G over $\mathfrak{F}$ plays a fundamental part. It is well-known that if $\mathfrak{F}$ has characteristic zero or a prime not dividing the order of G then this algebra is semisimple, and that in consequence the representations of G over $\mathfrak{F}$ are completely reducible.

“The central problem discussed in the dissertation is that of extending the theory to the case where the group G is replaced by a finite semigroup. Necessary and sufficient conditions are found for the semigroup algebra to be semisimple (with a restriction on the characteristic of $\mathfrak{F}$), and a study is made of the representation theory in the semisimple case. The results are then applied to certain important types of semigroups.”

Munn and semigroup algebras

Given a semigroup S, a series is a finite descending sequence of inclusions of the form

$$S = S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n \supset S_{n+1} = \emptyset,$$

where each S_i (except S_{n+1}) is a subsemigroup of S, and S_{i+1} is an ideal of S_i.

The factors of the series are the Rees quotients S_i/S_{i+1}.

A proper series is one in which all inclusions are strict.

A refinement of a series is any series that contains all the terms of the given series.

Two series are isomorphic if there is a one-one correspondence between their terms such that corresponding factors are isomorphic.

A refinement is proper if it is a proper series and contains strictly more terms than the original series.

A composition series is a proper series with no proper refinements.

Munn and semigroup algebras

Derived necessary and sufficient conditions for a semigroup to possess a composition series.

Similarly for principal series: proper series in which every term is an ideal of S, and which have no proper refinements with this property.

The factors of a principal series are termed principal factors.

A semigroup is semisimple if it has a principal series for which all the factors are simple.

Theorem: If M is an ideal of a semigroup S, then S is semisimple if and only if both M and S/M are semisimple.

Theorem: A semigroup is regular (inverse) if and only if all its principal factors are regular (inverse).

Munn and semigroup algebras

Let $S = \{s_1, \ldots, s_n\}$ be a finite semigroup and $\mathfrak{F}$ be a field.

The algebra $\mathfrak{A}_{\mathfrak{F}}(S)$ of S over $\mathfrak{F}$ is the associative linear algebra over $\mathfrak{F}$ with basis S and multiplication

$$\left(\sum_i \lambda_i s_i\right)\left(\sum_j \mu_j s_j\right) = \sum_{i,j} \lambda_i \mu_i s_i s_j,$$

where $\lambda_i, \mu_i \in \mathfrak{F}$.

Slightly more convenient to work with contracted semigroup algebra $\mathfrak{A}_{\mathfrak{F}}(S)/\mathfrak{A}_{\mathfrak{F}}(z)$, where z is the zero of S (if it exists) and $\mathfrak{A}_{\mathfrak{F}}(z)$ denotes the one-dimensional algebra over $\mathfrak{F}$ with basis $\{z\}$.

There is a one-one correspondence between the representations of $\mathfrak{A}_{\mathfrak{F}}(S)$ and those of $\mathfrak{A}_{\mathfrak{F}}(S)/\mathfrak{A}_{\mathfrak{F}}(z)$.

Munn and semigroup algebras

Introduce $M_{mn}[\mathfrak{A}, P]$, the algebra of all $m \times n$ matrices over a ring $\mathfrak{A}$, with the usual addition for matrices, but with multiplication $\circ$ carried out with the help of a fixed $n \times m$ 'sandwich matrix' P: for $A, B \in M_{mn}[\mathfrak{A}, P]$, $A \circ B = APB$.

Let $S_{mn}[G, P]$ denote the finite Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ with $I = \{1, \ldots, m\}$ and $\Lambda = \{1, \ldots, n\}$.

The contracted algebra of such a semigroup over a field $\mathfrak{F}$ may be regarded as a matrix algebra $M_{mn}[\mathfrak{A}(G), P]$, where $\mathfrak{A}(G)$ denotes the algebra of the structure group G.

Theorem: The algebra $M_{mn}[\mathfrak{A}, P]$ is semisimple if and only if

1. $\mathfrak{A}$ is semisimple, and
2. P is non-singular, in the sense that there exists an $m \times n$ matrix Q over $\mathfrak{A}$ such that either $PQ = I_n$ or $QP = I_m$.

Munn and semigroup algebras

Theorem: Let S be a finite semigroup, and let $\mathfrak{F}$ be a field of characteristic c. The semigroup algebra $\mathfrak{A}(S)$ of S over $\mathfrak{F}$ is semisimple if and only if

1. $c = 0$ or c does not divide the order of the structure group of any of the principal factors of S, and
2. each principal factor of S is a c-non-singular* simple or 0-simple semigroup.

*isomorphic to a Rees matrix semigroup of the form $S_{nn}[G, P]$, where the sandwich matrix P is non-singular as a matrix over the group algebra $\mathfrak{A}(G)$ over any field of characteristic c

Went on to build on Clifford's work by constructing irreducible representations of a finite 0-simple semigroup from those of its structure group.

J. S. Ponizovskii (1928–2012)

Ponizovskii and semigroup algebras

1956 МАТЕМАТИЧЕСКИЙ СБОРНИК Т. 38 (80), № 2

О матричных представлениях ассоциативных систем*

И. С. Понизовский (Кемерово)

В теории представлений групп очень важную роль играет теорема Машке о полной приводимости любого представления конечной группы над полем комплексных чисел, а также различные обобщения этой теоремы. Настоящая статья посвящена вопросу о том, в какой мере она переносится на представления конечных ассоциативных систем. Более точно в статье решается следующий вопрос:

Дано поле P; указать класс конечных ассоциативных систем, все представления которых матрицами с элементами из поля P вполне приводимы (такого рода конечные ассоциативные системы в дальнейшем для краткости называются P-системами).

Статья состоит из четырех параграфов. В § 1 излагаются необходимые сведения из теории ассоциативных систем; там же вводится основной инструмент исследования — системное кольцо, и излагаются его простейшие свойства, необходимые для дальнейшего. § 2 посвящен установлению критерия того, что данная конечная ассоциативная система является P-системой (теоремы 1 и 2). В § 3 даются некоторые приложения полученного критерия; в частности, показывается, что конечные системы частичных преобразований ведут себя, с точки зрения полной приводимости их представлений, как конечные группы. В § 4 строятся неприводимые представления произвольной P-системы. Поле при этом предполагается алгебраически замкнутым; это ограничение несущественно и вводится лишь для сокращения выкладок.

Автор выражает глубокую благодарность Е. С. Ляпину за предложенную задачу и ценные советы в процессе ее решения.

Мы употребляем следующие обозначения:

$\cup$, $\setminus$ — теоретико-множественные сумма и разность.

$\oplus$ — знак прямой суммы в кольце.

(a_{ij}) — матрица с элементами a_{ij}.

$\varphi(\mathfrak{N})$ — образ $\mathfrak{N} \subset \mathfrak{M}$ при отображении φ множества $\mathfrak{M}$.

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* После того как рукопись настоящей статьи была отослана в редакцию «Математического сборника», автору стала известна статья Мунна, опубликованная в Proc. of Cambr. Phil. Soc., **51**, № 1 (1955), 1–15. Статья Мунна содержит наиболее существенные предложения (теоремы 1 и 2), а также некоторые другие утверждения (теоремы 6 и 8) нашей статьи. Следует отметить, однако, что все результаты нашей статьи, в том числе и отсутствующие у Мунна теоремы 3, 4, 7, 10, 11, получены автором в 1952 г. и в июне 1953 г., за полтора года до опубликования статьи Мунна; по этим результатам автором была защищена кандидатская диссертация, тогда же был опубликован и автореферат диссертации, в котором были сформулированы все результаты настоящей статьи.

Ponizovskii and semigroup algebras

Studied *P-systems*: semigroups whose semigroup algebras are semisimple.

Theorem: A semigroup with a principal series is a P-system if and only if all principal factors are P-systems.

Conditions for a symmetric inverse semigroup to be a P-system.

Conditions for a Rees matrix semigroup to be a P-system.

Constructed all irreducible representations of a Rees matrix semigroup from those of its structure group.

Parallel developments

	1933: Sushkevich/ finite simple semigroups
1942: Clifford/ completely 0-simple semigroups	
1955: Munn/ broader theory	1956: Ponizovskii/ broader theory
1961 [1972]: Clifford and Preston/ presentation of Munn's theory	1960 [1963]: Lyapin/ nothing on representations