Cellular and standardly based semigroup algebras

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Semigroup Algebras

Definition

Let k be a field and let S be a finite semigroup. We define kS to be the k-algebra with basis $\{s \mid s \in S\}$ and multiplication given by

$$\sum_{s\in S} a_s s \sum_{t\in S} b_t t = \sum_{s\in S}\sum_{t\in S} a_s b_t (st).$$

Aim

Find a ‘nice’ basis of kS which gives us information about the representation theory of kS.

Algebras with nice bases

Cellular algebras (Graham & Lehrer (1996))

- Algebra with anti-involution $*$ and multiplication of basis elements expressed by a 'straightening formula'.
- Gives useful tools for understanding representation theory
  - Cell modules $\rightsquigarrow$ simple modules
  - Bilinear form $\rightsquigarrow$ test for semisimplicity.
  - Global dimension / quasi-hereditary: via Cartan determinants (König & Xi (1999), Xi (2003)).

Standardly based algebras (Du & Rui (1998))

- Generalises cellularity by removing the anti-involution condition.
- Still maintains many of the nice properties of cellular algebras e.g. bilinear form and cell modules.

Cellular and standardly based semigroup algebras

Inverse semigroups (East 2005)

kI_n is cellular, I_n - the symmetric inverse monoid of partial bijections

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & -\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & - & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ - & 1 & -\end{pmatrix}$$

Diagram semigroups (Wilcox 2007)

$k\mathcal{P}_n$ is cellular where $\mathcal{P}_n$ is the partition monoid.

Transformation semigroups (May 2015)

kT_n is not cellular, but is standardly based, where T_n is the full transformation monoid

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\end{pmatrix}$$

Cellular algebras

Definition (Graham & Lehrer (1996) - Sketch of definition)

A cellular algebra A over a field k is an algebra with a basis

$$\mathcal{C} = \{c_{st}^{\lambda} \mid \lambda \in \Lambda, s, t \in M(\lambda)\}$$

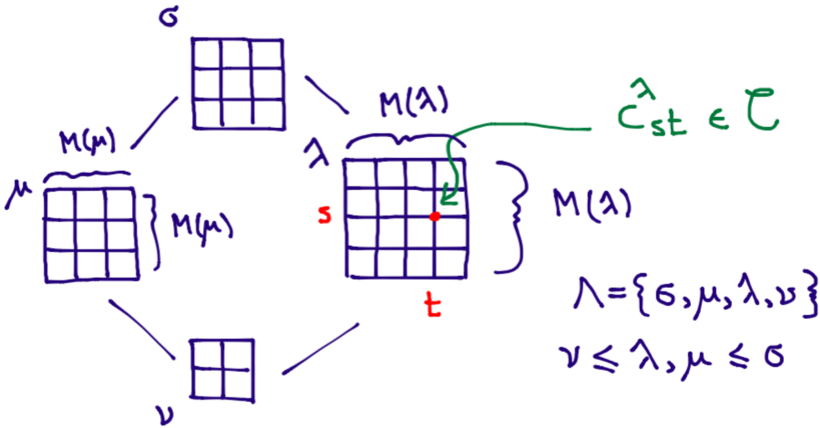
where

- Λ is a finite poset, $M(\lambda)$ is a finite index set for each $\lambda \in \Lambda$.
- The k-linear map $* : c_{st}^{\lambda} \mapsto c_{ts}^{\lambda}$ is an anti-involution of A

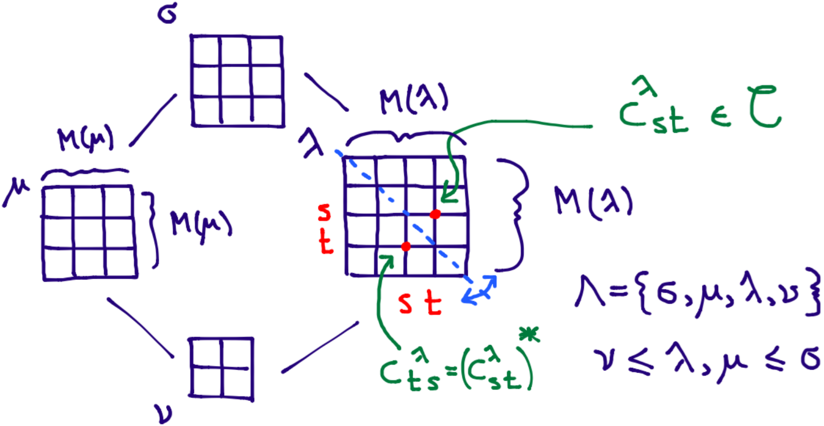
$$(a^*)^* = a \quad \text{and} \quad (ab)^* = b^* a^* \quad \text{for all } a, b \in A.$$

- If $a \in A$ and $c_{st}^{\lambda} \in \mathcal{C}$ then ac_{st}^{λ} has certain nice properties. (Using $*$, we have similar properties for $c_{st}^{\lambda}a$.)

Cellular basis picture



Cellular basis picture



Note: The anti-involution $* : c^\lambda_{st} \to c^\lambda_{ts}$ corresponds to reflecting each square by the main diagonal.

Example: kS_3 is cellular – Murphy (1992)

$\lambda = (1,1,1)$

	$\begin{matrix}1\\2\\3\end{matrix}$
$\begin{matrix}1\\2\\3\end{matrix}$	I

$\lambda = (2,1)$

	$\begin{matrix}1 & 2\\3 & \end{matrix}$	$\begin{matrix}1 & 3\\2 & \end{matrix}$
$\begin{matrix}1 & 2\\3 & \end{matrix}$	$I + (1\ 2)$	$(I + (1\ 2))(2\ 3)$
$\begin{matrix}1 & 3\\2 & \end{matrix}$	$(2\ 3)(I + (1\ 2))$	$(2\ 3)(I + (1\ 2))(2\ 3)$

$\lambda = (3)$

	$\begin{matrix}1 & 2 & 3\end{matrix}$
$\begin{matrix}1 & 2 & 3\end{matrix}$	$I + (1\ 2) + (1\ 3) + (2\ 3) + (1\ 2\ 3) + (1\ 3\ 2)$

$\Lambda = \mathcal{P}_3$ - partitions of 3, ordered by $(3) < (2,1) < (1,1,1)$
$M(\lambda) = \mathrm{Std}(\lambda)$ - standard λ-tableaux
$* =$ map induced by $^{-1} : S_3 \to S_3$

Semigroup and group algebras

The Murphy basis can be used to show in general:

Proposition

kS_n is a cellular algebra.

General question

Which semigroup algebras kS are cellular?

Main idea

Prove results which relate:

$$\text{cellularity of } kS \longleftrightarrow \text{cellularity of } kH_i \ (i \in I)$$

where $\{H_i \ (i \in I)\}$ is the set of maximal subgroups of S.

Green's relations and maximal subgroups

Green's relations: equivalence relations reflecting ideal structure.

For $u, v \in S$ we define

$$u\mathcal{R}v \;\Leftrightarrow\; uS \cup \{u\} = vS \cup \{v\}, \quad u\mathcal{L}v \;\Leftrightarrow\; Su \cup \{u\} = Sv \cup \{v\},$$

$$\mathcal{H} = \mathcal{R} \cap \mathcal{L}.$$

{ Maximal subgroups of S } = { $\mathcal{H}$-classes that contain idempotents }

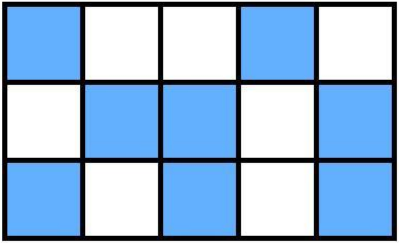
Example. Let $S = T_3$ and $\epsilon = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right) \in E(S)$. Then

$$H_\epsilon = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{smallmatrix}\right) \right\} \cong S_2.$$

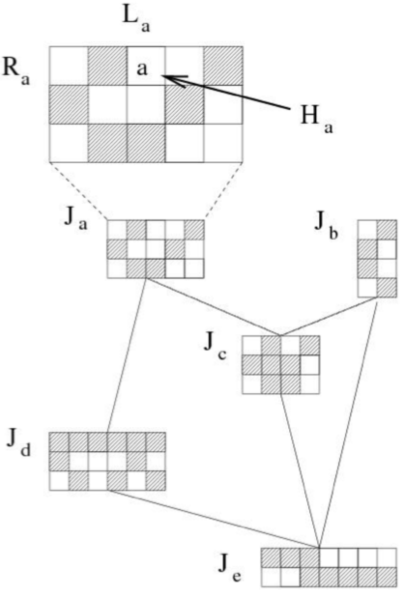
Regular $\mathcal{D}$-classes

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

- A $\mathcal{D}$-class is (von Neumann) regular if it contains an idempotent
- A regular $\mathcal{D}$-class has ≥ 1 idempotent in every $\mathcal{R}$- and every $\mathcal{L}$-class.
- All maximal subgroups in a regular $\mathcal{D}$-class are isomorphic.



Structure of a finite regular semigroup



S - semigroup, $x, y \in S$

$$x\mathcal{R}y \Leftrightarrow xS^1 = yS^1$$
$$x\mathcal{L}y \Leftrightarrow S^1x = S^1y$$
$$x\mathcal{J}y \Leftrightarrow S^1xS^1 = S^1yS^1$$

- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} (= \mathcal{J})$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $J_x \leq J_y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1$

Inverse semigroups

Definition

S is inverse if for all $s \in S$ there is a unique $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$.

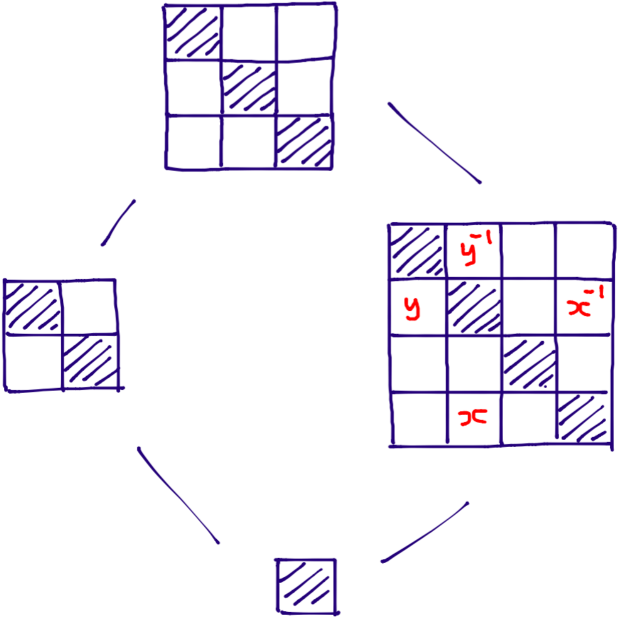
Equivalently S is inverse $\Leftrightarrow$ every $\mathcal{R}$- and $\mathcal{L}$-class contains exactly one idempotent.

Example

The symmetric inverse semigroup I_n

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & - & 2 \end{pmatrix}$$

Inverse semigroup structure



Cellular inverse semigroup algebras

Theorem (East 2005)

If S is a finite inverse semigroup and all maximal subgroups of S are cellular[1] *then kS is a cellular algebra. The basis elements are*

$$u_L^{-1} \cdot c_{st}^{\lambda} \cdot u_K$$

where

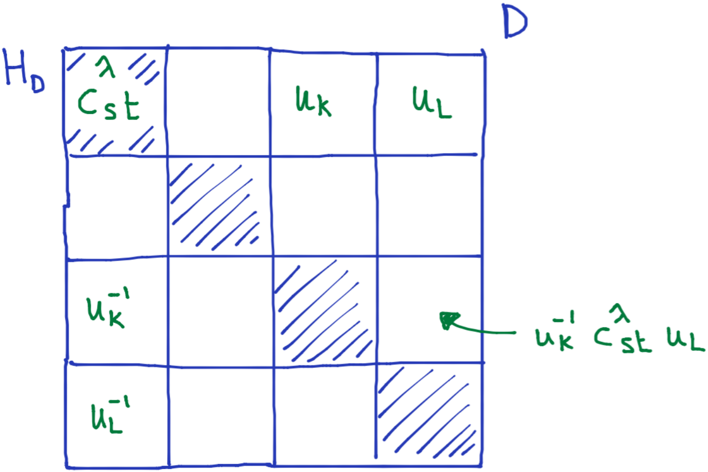
- c_{st}^{λ} *is an element of a cellular basis of the cellular algebra* kH_D
- u_L, u_K *are* $\mathcal{L}$*-class representatives in the* $\mathcal{R}$*-class of* H_D.

Poset Λ for kS is given by taking a 'product' of the poset $(\mathcal{D}, \leq)$ with the Λ_D posets.

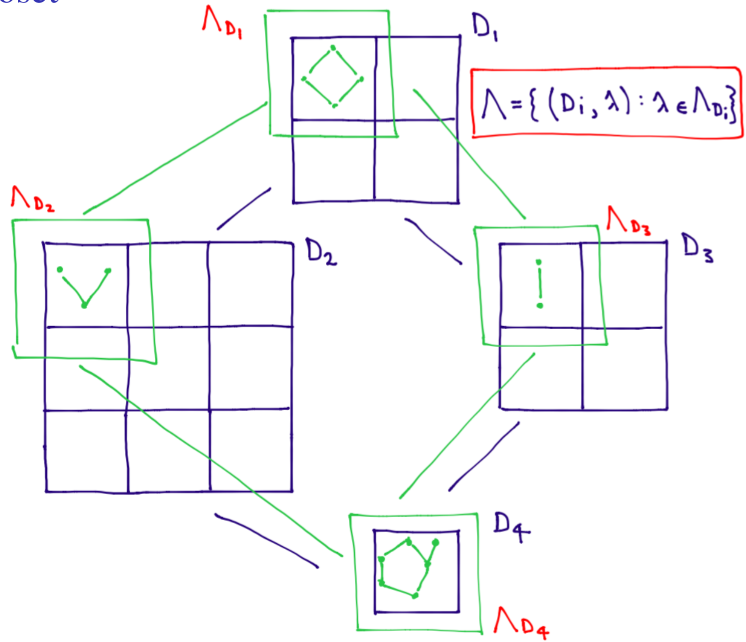
The cells $M(D, \lambda) \times M(D, \lambda)$ are given by taking a 'product' of the square $M(\lambda) \times M(\lambda)$ cells for kH_D with the square $\mathcal{D}$-classes.

[1](and the anti-involutions $*$ for these cellular structures are suitably compatible)

The basis elements

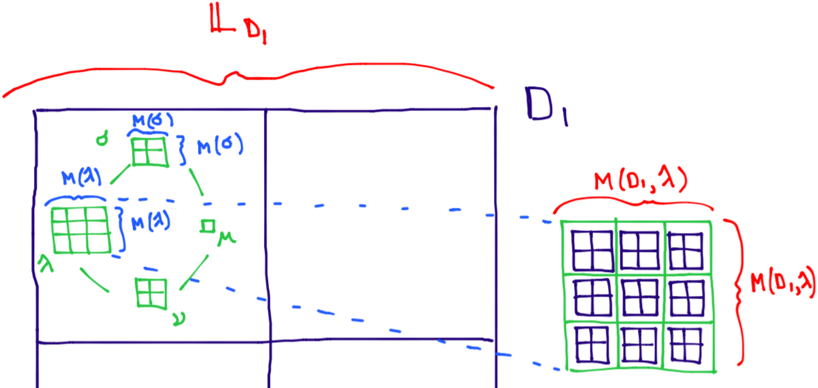


The poset



Λ_{D_1}
D_1
$\Lambda = \{ (D_i, \lambda) : \lambda \in \Lambda_{D_i} \}$
Λ_{D_2}
D_2
Λ_{D_3}
D_3
D_4
Λ_{D_4}

The cells



$$M(D_1,\lambda) = \mathbb{L}_{D_1} \times M_{D_1}(\lambda)$$

$$C = \{ u_L^{-1} \; \overset{\lambda}{c}_{st} \; u_K \}$$

The symmetric inverse monoid algebra kI_n

Theorem (East 2005)

If S is a finite inverse semigroup and all the maximal subgroups of S are cellular then kS is a cellular algebra.

Corollary (East (2005))

kI_n is a cellular algebra.

Proof: $\{S_r : 1 \leq r \leq n\}$ are the maximal subgroups of I_n and the symmetric group algebras kS_r are all cellular.

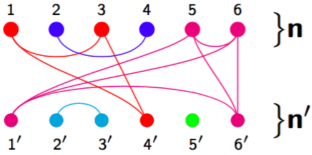
Diagram semigroups

The partition monoid is

$$\begin{aligned}\mathcal{P}_n &= \{ \text{ set partitions of } \{1,\ldots,n\} \cup \{1',\ldots,n'\} \ \} \\ &= \{ \text{ eq. classes of graphs on } \{1,\ldots,n\} \cup \{1',\ldots,n'\} \ \}.\end{aligned}$$

Example

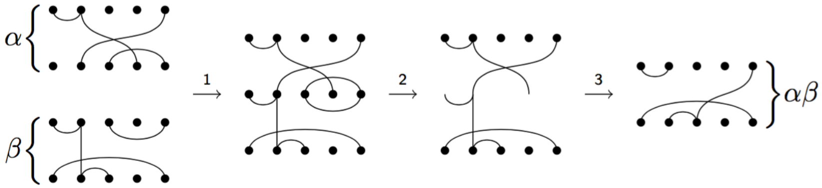
$$\alpha = \Big\{ \{1,3,4'\}, \{2,4\}, \{5,6,1',6'\}, \{2',3'\}, \{5'\} \Big\} \in \mathcal{P}_6$$



Partition monoid multiplication

Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

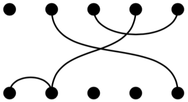
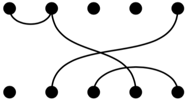
1. connect bottom of α to top of β;
2. remove middle vertices and floating components.



The operation is associative so $\mathcal{P}_n$ is a monoid.

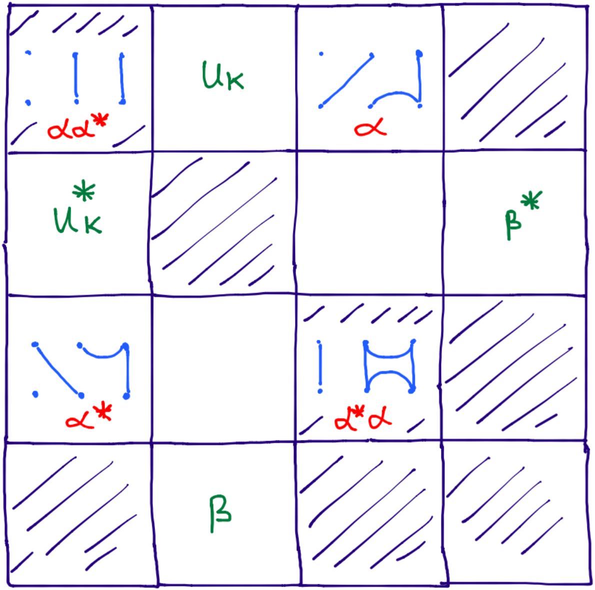
Properties of the partition monoid $\mathcal{P}_n$

There is an anti-involution operation 'vertical flip' $* : \mathcal{P}_n \to \mathcal{P}_n$:

$\alpha =$  $\alpha^* =$ 

- $*$ interchanges $\mathcal{R}$- and $\mathcal{L}$-classes $\Rightarrow$ $\mathcal{D}$-classes are square.
- Maximal subgroups of $\mathcal{P}_n$ are $\{S_r : 1 \leq r \leq n\}$.
- Each $\mathcal{R}$-class and $\mathcal{L}$-class contain a unique projection. (Projection = an idempotent α such that $\alpha^* = \alpha$.)

Partition monoid $\mathcal{D}$-class structure



Cellular diagram algebras

Theorem (Wilcox 2007)

If S is a finite regular semigroup with an anti-involution $* : S \to S$ and all maximal subgroups of S are cellular[2] *then kS is a cellular algebra. The basis elements are*

$$u_L^* \cdot c_{st}^\lambda \cdot u_K$$

where

- c_{st}^λ *is an element of a cellular basis of the cellular algebra kH_D*
- u_L, u_K *are $\mathcal{L}$-class representatives in the $\mathcal{R}$-class of H_D.*

Actually, Wilcox proved more general results about cellularity of 'twisted' semigroup algebras $k^\alpha[S]$, which allowed him to recover:

Corollary (Wilcox 2007)

The partition, Temperley–Lieb, & Brauer algebras are all cellular.

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[2](and each $\mathcal{D}$-class has an idempotent $e_D \in H_D$ fixed by $*$)

Structure of T_n

Green's relations

$\alpha, \beta \in T_n$
$\alpha \mathcal{L} \beta \Leftrightarrow \mathrm{im}\alpha = \mathrm{im}\beta$
$\alpha \mathcal{R} \beta \Leftrightarrow \ker \alpha = \ker \beta$
$\alpha \mathcal{D} \beta \Leftrightarrow |\mathrm{im}\alpha| = |\mathrm{im}\beta|$

Maximal subgroups of T_n are:

$$\{S_r : 1 \leq r \leq n\}$$

$\mathcal{D}$-classes are not square.
There is no natural anti-involution.

kT_n is not a cellular algebra.

* S_4

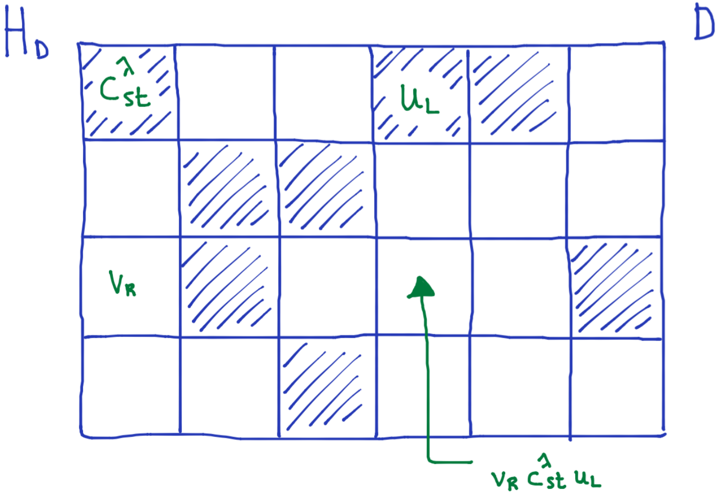
	123	124	134	234
12|3|4			*	*
13|2|4		*		*
14|2|3	*			*
23|1|4		*	*	
24|1|3	*		*	
34|1|2	*	*		

	12	13	14	23	24	34
123|4			*		*	*
124|3		*		*		*
134|2	*			*	*	
234|1	*	*	*			
12|34		*	*	*	*	
13|24	*		*	*		*
14|23	*	*			*	*

	1	2	3	4
1234	*	*	*	*

The basis elements

...if we tried to build a cellular basis for T_n



Standardly based algebras

Definition (Du & Rui (1998) - Sketch of definition)

A standardly based algebra A over a field k is an algebra with a basis

$$\mathcal{C} = \{c_{st}^{\lambda} \mid \lambda \in \Lambda, s \in \mathcal{I}(\lambda), t \in \mathcal{J}(\lambda)\}$$

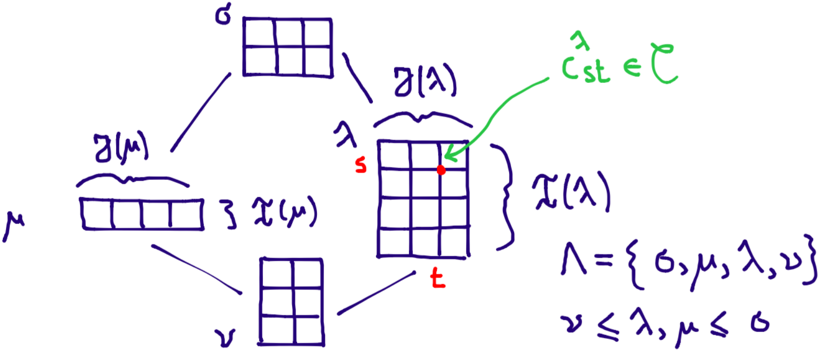
such that

- Λ is a finite poset, $\mathcal{I}(\lambda)$ & $\mathcal{J}(\lambda)$ are finite index sets
- If $a \in A$ and $c_{st}^{\lambda} \in \mathcal{C}$ then ac_{st}^{λ} and $c_{st}^{\lambda}a$ have certain nice properties.

Remark

- cellular $\Rightarrow$ standardly based (but not conversely).
- In 2015, May defined the notion of a 'cell algebras'. Cell algebras coincide with standardly based algebras.

Standard basis picture



Standardly based semigroup algebras

Theorem (May 2015)

If S is a finite regular semigroup and all maximal subgroups of S are standardly based then kS is standardly based. The basis elements are of the form

$$v_R c_{st}^{\lambda} u_L$$

where

- c_{st}^{λ} *is an element of a standard basis of a standardly based algebra* kH_D.
- v_R *is an* $\mathcal{R}$*-class representative.*
- u_L *is a* $\mathcal{L}$*-class representative.*

Corollary (May (2015))

kT_n is a standardly based algebra.