Computing maximal subsemigroups of finite semigroups

Wilf Wilson

18^{th} March 2016

What is a maximal subsemigroup?

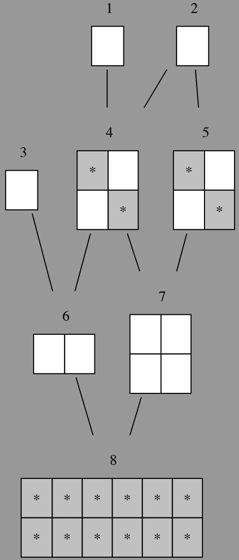
Definition (Maximal subsemigroup)

A maximal subsemigroup is a proper subsemigroup which is not contained in another proper subsemigroup.

Examples of semigroups and their maximal subsemigroups

- $\varnothing$ is a maximal subsemigroup of the trivial semigroup.
- The maximal subsemigroups of a non-trivial finite group are its maximal subgroups.
- $T_n \setminus \{\text{maps of rank } n-1\}$ is a maximal subsemigroup of T_n.
- $\big((1,\infty),\times\big)$ has no maximal subsemigroups.

Multiplication in a finite semigroup



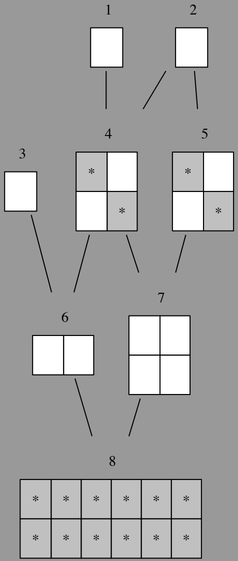
A visual interpretation is useful.

S is finite, so $\mathscr{D} = \mathscr{J}$.

Facts about Green's $\mathscr{J}$ relation:

(a) $x \mathscr{J} y$ iff $S^1 x S^1 = S^1 y S^1$,
(b) $J_x \leq J_y$ iff $S^1 x S^1 \subseteq S^1 y S^1$,
(c) $J_{xy} \leq J_x$ and $J_{xy} \leq J_y$.

The form of a maximal subsemigroup

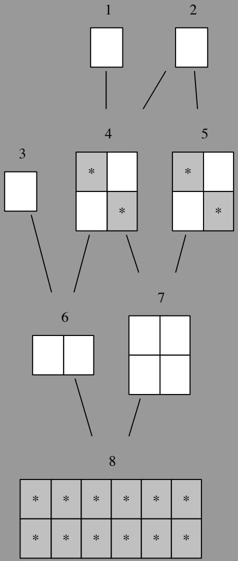


A maximal subsemigroup lacks part of precisely **one** $\mathscr{J}$-class.

The remaining part of it either:

(1) contains part of *every* $\mathscr{H}$-class;
(2) is a union of rows and columns;
(3) is a union of only rows;
(4) is a union of only columns;
(5) is empty.

The rough idea of our algorithm

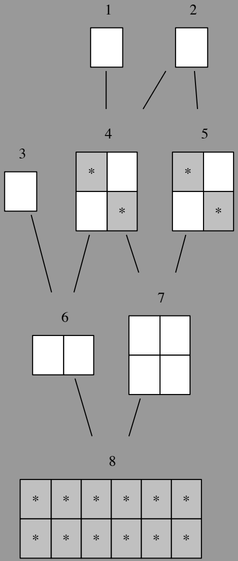


Go through each $\mathscr{J}$-class in turn.

Work out which maximal subsemigroups (if any) arise by removing parts of that $\mathscr{J}$-class.

(Non-regular $\mathscr{J}$-classes are easy).

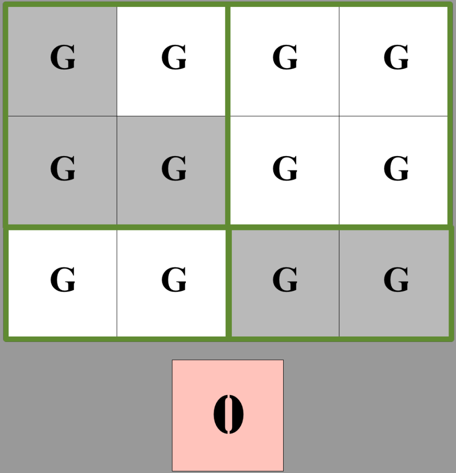
Some of the problems to consider



Foremost, a maximal subsemigroup is a **subsemigroup**. Hence, for a maximal subsemigroup M with partially-missing $\mathscr{J}$-class J:

- multiplication within $M \cap J$ is closed.
- $M \cap J$ contains the elements generated outside of J.
- $M \cap J$ is closed under multiplication by other elements.

Rees 0-matrix semigroups



Rees 0-matrix semigroups

$g_1^{-1}Vg_1$	$g_1^{-1}Vg_1$	$g_1^{-1}Vg_2$	$g_1^{-1}Vg_2$
$g_1^{-1}Vg_1$	$g_1^{-1}Vg_1$	$g_1^{-1}Vg_2$	$g_1^{-1}Vg_2$
$g_2^{-1}Vg_1$	$g_2^{-1}Vg_1$	$g_2^{-1}Vg_2$	$g_2^{-1}Vg_2$

0

(1): $M \cap J$ intersects every $\mathscr{H}$-class of J

- Define $E(J)$ to be the set of idempotents of J.
- Define X to be the set of generators *above* J.

Let M be a subset which intersects every $\mathscr{H}$-class of J.

Theorem

M is a maximal subsemigroup if and only if $M \cap J$ is a maximal subsemigroup of the principal factor of J which contains $E(J)X \cap J$.

Digraphs.

For the other types, we create some digraphs and reduce the search for maximal subsemigroups to a search within these digraphs.

(2): $M \cap J$ is a union of both rows and columns J

Let M be a subset such that $M \cap J$ is a union of rows and columns.

Theorem

M is a maximal subsemigroup if and only if

- *the rows are a union of vertices of $\Gamma_{\mathscr{R}}$ with no out-neighbours;*
- *the columns are a union of vertices of $\Gamma_{\mathscr{L}}$ with no out-neighbours;*
- *these vertices correspond to a maximal independent set of Δ;*
- *every edge of Δ' is incident to one of these vertices.*

(3): $M \cap J$ is a union of rows of J

Let M be a subset such that $M \cap J$ is a union of rows only.

Theorem

M is a maximal subsemigroup if and only if:

- *M is not contained in a maximal subsemigroup of type (2);*
- *the missing rows form a vertex of $\Gamma_{\mathscr{R}}$;*
- *that vertex has no in-neighbours;*
- *that vertex is not red.*

The theorem for maximal subsemigroups of type (4) is dual.

(5): $M \cap J = \varnothing$

Theorem

A maximal subsemigroup can be formed by removing J if and only if

- *J isn't generated by the rest of the semigroup, and*
- *there are no maximal subsemigroups of types (1) to (4).*

Simplified summary of the algorithm

Work out all of the information contained in the semigroup diagram:

- the Green's relations ($\mathscr{J}$, $\mathscr{R}$, $\mathscr{L}$, and $\mathscr{H}$);
- the $\mathscr{J}$-class partial order;
- the location of the generators;
- the location of idempotents.

Go through each $\mathscr{J}$-class in turn:

- If it is a maximal $\mathscr{J}$-class, calculate the maximal subsemigroups of the principal factor; otherwise
- Construct the necessary digraphs;
- Search these digraphs for various graph-theoretical properties to find maximal subsemigroups.

Upcoming functionality in the Semigroups package

The `MaximalSubsemigroups` function.

What it will be possible to find:

(a) All maximal subsemigroups,
(b) Maximal subsemigroups which contain a given set of elements,
(c) Maximal subsemigroups which lack part of a given $\mathscr{J}$-class.

How the answers can be returned:

(a) As a list of GAP semigroup objects,
(b) As a list of generating sets,
(c) As a number (only count them, don't create them).