Two Kinds of Congruence Networks on Regular Semigroups

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NBSAN, St Andrews, March 18, 2016

Outline

1. Notation & terminology
2. $\mathcal{TK}$-network on inverse semigroups
3. $\mathcal{TV}$-networks on regular semigroups
4. other work

Terminology

$a \in S$ is regular

— $(\exists x \in S)\ axa = a$

- regular semigroups

  — all elements are regular

- inverse semigroup

  — a regular semigroup whose idempotents commute

- congruence

  — compatible equivalence relation, i.e.

  $(\forall s, t, s', t' \in S)\ [(s, t) \in \rho \text{ and } (s', t') \in \rho] \Rightarrow (ss', tt') \in \rho$

- $\mathcal{P}$-type congruence

  — S/ρ is a $\mathcal{P}$-type semigroup

Congruence

- kernel-trace approach

  Let ρ be a congruence on S,

  $\operatorname{tr}\rho = \rho|_{E(S)}$, $\quad$ $\operatorname{Ker}\rho = \{x \in S \mid (\exists e \in E(S))\, x \,\rho\, e\}$.

- inverse semigroup

  1974, Scheiblich $\quad$ $\rho = \rho_{(\operatorname{tr}\rho, \operatorname{Ker}\rho)}$

  1978, Petrich $\quad$ congruence pair

Congruence pair

[1978, Petrich]

Definition

The pair (K, τ) is a *congruence pair* for S if K is a normal subsemigroup of S, τ is a normal congruence on $E(S)$, and these two satisfy:

(i) $ae \in K,\ e\,\tau\, a^{-1}a \Rightarrow a \in K \quad (a \in S, e \in E(S))$;

(ii) $k \in K \Rightarrow kk^{-1}\,\tau\, k^{-1}k$.

In such a case, define a relation $\rho_{(K,\tau)}$ on S by

$$a\,\rho_{(K,\tau)}\,b \iff a^{-1}a\,\tau\,b^{-1}b,\ ab^{-1} \in K.$$

Theorem

Let S be an inverse semigroup. If (K, τ) is a congruence pair for S, then $\rho_{(K,\tau)}$ is the unique congruence ρ on S for which $\operatorname{Ker}\rho = K$ *and* $\operatorname{tr}\rho = \tau$. *Conversely, if ρ is a congruence on S, then* $(\operatorname{Ker}\rho, \operatorname{tr}\rho)$ *is a congruence pair for S and $\rho_{(K,\tau)} = \rho$.*

Characterization of congruences

Inverse semigroup		Regular semigroup
1974, Scheiblich	$\rho = \rho_{(\operatorname{tr}\rho, \operatorname{Ker}\rho)}$	1979, Feigenbaum
1978, Petrich	congruence pair	1986, Pastijn – Petrich

Congruence pair

Definition

A pair (K,τ) is a *congruence pair* for S if

(i) K is a normal subset of S,

(ii) τ is a normal equivalence on $E(S)$,

(iii) $K \subseteq \operatorname{Ker}(\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^\flat$,

(iv) $\tau \subseteq \operatorname{tr}\pi_K$.

In such a case, we define

$$\rho_{(K,\tau)} = \pi_K \cap (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^\flat.$$

Theorem

Let S be a regular semigroup. If (K,τ) is a congruence pair for S, then $\rho_{(K,\tau)}$ is the unique congruence ρ on S for which $\operatorname{Ker}\rho = K$ and $\operatorname{tr}\rho = \tau$. Conversely, if ρ is a congruence on S, then $(\operatorname{Ker}\rho, \operatorname{tr}\rho)$ is a congruence pair for S and $\rho = \rho_{(K,\tau)}$.

Congruence triple

[1986, Pastijn – Petrich]

Definition

A triple (γ, K, δ) consisting of normal equivalences $\gamma \in \mathcal{E}(S/\mathcal{L})$ and $\delta \in \mathcal{E}(S/\mathcal{R})$ and a normal subset $K \subseteq S$, is a *congruence triple* if

(i) $\overline{\gamma} = (\overline{\gamma} \cap \overline{\delta})^\flat \vee \mathcal{L}$, $\overline{\delta} = (\overline{\gamma} \cap \overline{\delta})^\flat \vee \mathcal{R}$;

(ii) $K \subseteq \operatorname{Ker} \overline{\gamma}^\flat$, $\overline{\gamma} \subseteq \theta_K^\flat \vee \mathcal{L}$;

(iii) $K \subseteq \operatorname{Ker} \overline{\delta}^\flat$, $\overline{\delta} \subseteq \theta_K^\flat \vee \mathcal{R}$.

If this is the case, we define

$$\rho_{(\gamma,K,\delta)} = (\overline{\gamma} \cap \theta_K \cap \overline{\delta})^\flat.$$

Theorem

Let S be a regular semigroup. If (γ, K, δ) is a congruence triple for S, then $\rho_{(\gamma,K,\delta)}$ is the unique congruence ρ on S such that γ is the $\mathcal{L}$-part of ρ, $K = \operatorname{Ker} \rho$ and δ is the $\mathcal{R}$-part of ρ. Conversely, if ρ is a congruence on S, then $(\gamma, K, \delta) = ((\rho \vee \mathcal{L})/\mathcal{L}, \operatorname{Ker} \rho, (\rho \vee \mathcal{R})/\mathcal{R})$ is a congruence triple for S and $\rho = \rho_{(\gamma,K,\delta)}$.

Congruence

- kernel-trace approach

  Let ρ be a congruence on S,

  $$\operatorname{tr}\rho = \rho|_{E(S)}, \qquad \operatorname{Ker}\rho = \{x \in S \mid (\exists e \in E(S))\, x \,\rho\, e\}.$$

  $$\rho = \rho_{(\operatorname{tr}\rho, \operatorname{Ker}\rho)}.$$

- $\mathcal{T}$, $\mathcal{K}$-relation

  Let $\rho, \theta \in \mathcal{C}(S)$,

  $$\rho\,\mathcal{T}\,\theta \iff \operatorname{tr}\rho = \operatorname{tr}\theta, \qquad \rho\,\mathcal{K}\,\theta \iff \operatorname{Ker}\rho = \operatorname{Ker}\theta,$$

  $$\rho\,\mathcal{U}\,\theta \iff \rho \cap \leq = \theta \cap \leq, \qquad \rho\,\mathcal{V}\,\theta \iff \rho\,\mathcal{U}\,\theta \text{ and } \rho\,\mathcal{K}\,\theta,$$

  where $\leq$ is the natural partial order on $E(S)$.

  - $\mathcal{T} \cap \mathcal{K} = \varepsilon_{\mathcal{C}(S)} = \mathcal{T} \cap \mathcal{V}$

Definition

A triple (γ, π, δ) consisting of normal equivalences $\gamma \in \mathcal{E}(S/\mathcal{L})$ and $\delta \in \mathcal{E}(S/\mathcal{R})$ and a $\mathcal{V}$-normal congruence π on S, is a *$\mathcal{VT}$-congruence triple* if

(i) $\overline{\gamma} = (\overline{\gamma} \cap \overline{\delta})^\flat \vee \mathcal{L}$, $\overline{\delta} = (\overline{\gamma} \cap \overline{\delta})^\flat \vee \mathcal{R}$;

(ii) $\pi \subseteq (\overline{\gamma}^\flat)^V$, $\overline{\gamma} \subseteq \pi \vee \mathcal{L}$;

(iii) $\pi \subseteq (\overline{\delta}^\flat)^V$, $\overline{\delta} \subseteq \pi \vee \mathcal{R}$.

If this is the case, we define

$$\rho_{(\gamma,\pi,\delta)} = (\overline{\gamma} \cap \pi \cap \overline{\delta})^\flat.$$

Theorem

Let S be a regular semigroup. If (γ, π, δ) is a $\mathcal{VT}$-congruence triple for S, then $\rho_{(\gamma,\pi,\delta)}$ is the unique congruence ρ on S such that γ is the $\mathcal{L}$-part of ρ, π is the $\mathcal{V}$-part of ρ and δ is the $\mathcal{R}$-part of ρ. Conversely, if ρ is a congruence on S, then $(\gamma, \pi, \delta) = ((\rho \vee \mathcal{L})/\mathcal{L}, \overline{\mathcal{V}_{S/\rho}}^\flat, (\rho \vee \mathcal{R})/\mathcal{R})$ is a congruence triple for S and $\rho = \rho_{(\gamma,\pi,\delta)}$.

- $\rho\,\mathcal{T}\,\theta \iff \operatorname{tr}\rho = \operatorname{tr}\theta$, $\quad \rho\,\mathcal{K}\,\theta \iff \operatorname{Ker}\rho = \operatorname{Ker}\theta$,
  $\rho\,\mathcal{U}\,\theta \iff \rho \cap \leq = \theta \cap \leq$, $\quad \mathcal{V} = \mathcal{U} \cap \mathcal{K}$.

Result

For any $\rho \in \mathcal{C}(S)$, $\rho\mathcal{T} = [\rho t,\, \rho T]$, $\rho\mathcal{K} = [\rho k,\, \rho K]$, $\rho\mathcal{U} = [\rho u,\, \rho U]$,

$\rho\mathcal{V} = [\rho v,\, \rho V]$, *where*

$$\rho t = (\operatorname{tr}\rho)^{\sharp}, \qquad \rho T = \overrightarrow{\mathcal{H}_{S/\rho}}^{\flat},$$

$$\rho k = \{(x, x^2) \in S \times S \mid x \in \operatorname{Ker}\rho\}^{\sharp}, \qquad \rho K = \theta^{\flat}_{\operatorname{Ker}\rho},$$

$$\rho u = (\rho \cap \leq)^{\sharp}, \qquad \rho U = \overrightarrow{\mathscr{U}_{S/\rho}}^{\flat},$$

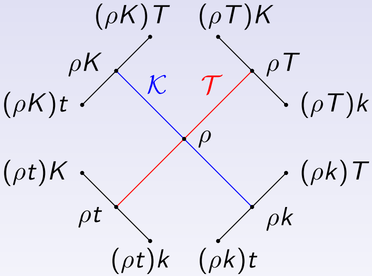
$$\rho v = \rho_U \vee \rho_K, \qquad \rho V = \rho U \cap \rho K = \overrightarrow{\mathscr{V}_{S/\rho}}^{\flat}.$$

Congruence

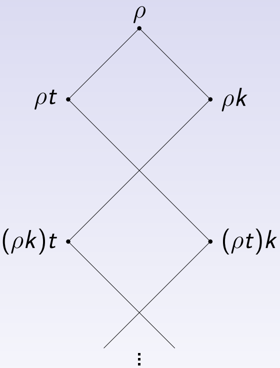
- kernel-trace approach
- $\mathcal{T}$, $\mathcal{K}$-relation
- congruence networks
  - single out various classes of semigroups of particular interest
  - structure

Congruence network

$\mathcal{T} \cap \mathcal{K} = \varepsilon$



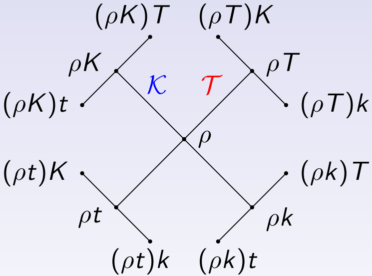
$\mathcal{TK}$-network of ρ



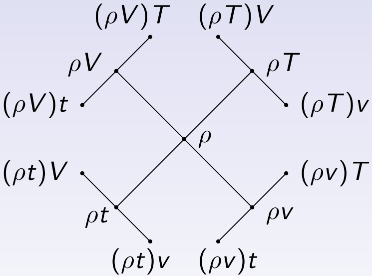
$\mathcal{TK}$-min network of ρ

Congruence network

$$\mathcal{T} \cap \mathcal{K} = \varepsilon \qquad \mathcal{T} \cap \mathcal{V} = \varepsilon$$

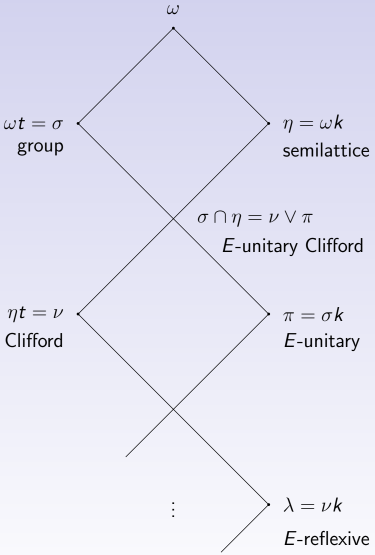


$\mathcal{TK}$-network of ρ



$\mathcal{TV}$-network of ρ

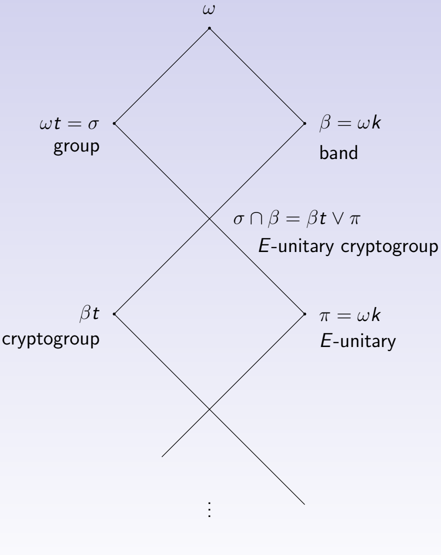
Inverse semigroup



$\mathcal{TK}$ min-network of ω

[1982, Petrich – Reilly]

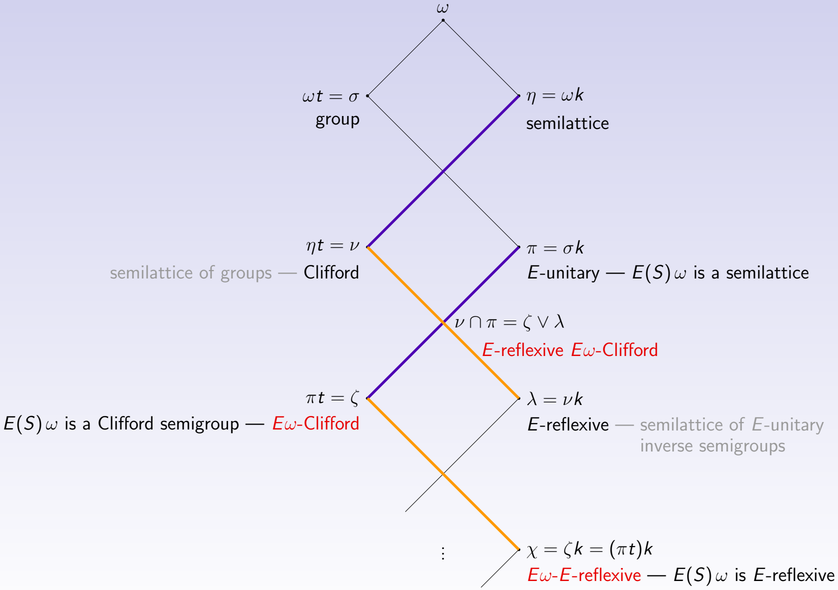
Regular semigroup



$\mathcal{TK}$ min-network of ω

[1988, Alimpić – Krgović]

$\mathcal{TK}$-network on inverse semigroup



$\mathcal{TK}$ min-network of ω

$E\omega$-Clifford semigroup and $E\omega$-E-reflexive semigroup

Proposition

The following conditions on an inverse semigroup S are equivalent.
(1) *S is an $E\omega$-Clifford semigroup;*
(2) *$\sigma \cap \mathcal{L}$ is a congruence;*
(3) *$\sigma \cap \mathcal{R}$ is a congruence;*
(4) *$\sigma \cap \mathcal{L} = \sigma \cap \mathcal{R}$;*
(5) *$\sigma \cap \mathcal{L} = \sigma \cap \mu$;*
(6) *there exists an idempotent separating E-unitary congruence on S;*
(7) *$\pi \subseteq \mu$;*
(8) *$\pi t = \varepsilon$;*
(9) *$e\sigma$ is a Clifford semigroup for every $e \in E(S)$,;*
(10) *S satisfies the implication $xy = x \Rightarrow y \in E(S)\zeta$;*
(11) *$E(S)\omega \subseteq E(S)\zeta$;*
(12) *$\pi \cap \mathcal{F} = \varepsilon$.*

Theorem

The following conditions on an inverse semigroup S are equivalent.
(1) *S is $E\omega$-E-reflexive;*
(2) *$\pi \cap \mathcal{F}$ is a congruence;*
(3) *$\pi \cap \mathcal{C}$ is a congruence;*
(4) *$\pi \cap \mathcal{F} = \pi \cap \tau$;*
(5) *$\pi \cap \mathcal{C} = \pi \cap \tau$;*
(6) *there exists an idempotent pure $E\omega$-Clifford congruence on S;*
(7) *$\zeta \subseteq \tau$;*
(8) *$\zeta k = \varepsilon$;*
(9) *$e\pi$ is E-unitary for every $e \in E(S)$;*
(10) *S satisfies the implication $xy = x, x \pi y \Rightarrow y \in E(S)$;*
(11) *$\zeta \cap \mathcal{L} = \varepsilon$.*

$E\omega$-Clifford congruence and $E\omega$-E-reflexive congruence

Proposition

The following statements concerning a congruence ρ on an inverse semigroup S are equivalent.

(1) ρ is an $E\omega$-Clifford congruence;

(2) $\pi_\rho \subseteq \rho T$, where π_ρ is the least E-unitary congruence on S containing ρ;

(3) $\operatorname{tr} \pi_\rho = \operatorname{tr} \rho$.

Proposition

The following statements concerning a congruence ρ on an inverse semigroup S are equivalent.

(1) ρ is $E\omega$-E-reflexive;

(2) $\zeta_\rho \subseteq \rho K$, where ζ_ρ is the least $E\omega$-Clifford congruence on S containing ρ;

(3) $\operatorname{Ker} \zeta_\rho = \operatorname{Ker} \rho$.

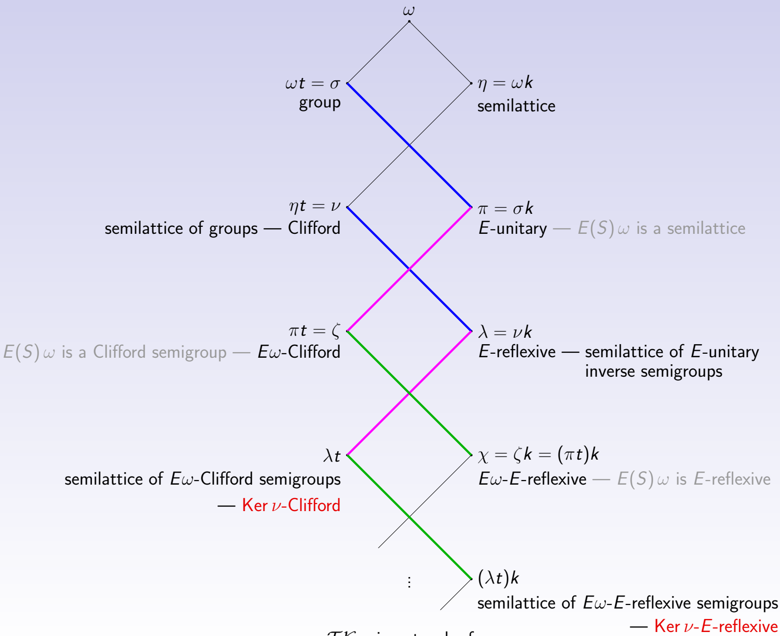
Coincidences

Petrich, *Inverse Semigroups*, Table III.8.10

	ω	σ	η	ν	π	λ	ζ	χ	μ	τ
σ	$E\omega = S$									
η	no c. pr. ideals	$E\omega = S$, no c. pr. ideals								
ν	$\sigma = \eta = \omega$	no c. pr. ideals	$E_A\omega = A$ ($\forall$ η-cl. A)							
π	$\sigma = \eta = \omega$	tr $\pi = \omega$	$E\omega = S$							
λ	$\sigma = \eta = \omega$	tr $\pi = \omega$	$E_A\omega = A$ ($\forall$ η-cl. A)							
ζ	$\sigma = \eta = \omega$	tr $\pi = \omega$	$E_A\omega = A$ ($\forall$ η-cl. A)							
χ	$\sigma = \eta = \omega$	tr $\pi = \omega$	$E_A\omega = A$ ($\forall$ η-cl. A)							
μ	group	trivial	Clifford	semil.	$E\omega = E\zeta$ and tr $\pi = \varepsilon$					
τ	semil.	E-un.	trivial	E-refl. tr π = tr η	E-un. E-disj.	E-refl. tr τ = tr λ	$E\omega$-E-refl. tr τ = tr π	E-disj. $E\omega$-E-refl.	E-disj. antig.	
ε	trivial	group	semil.	Clifford	E-un.	E-refl.	$E\omega$-Clifford	$E\omega$-E-refl.	antig.	E-disj.

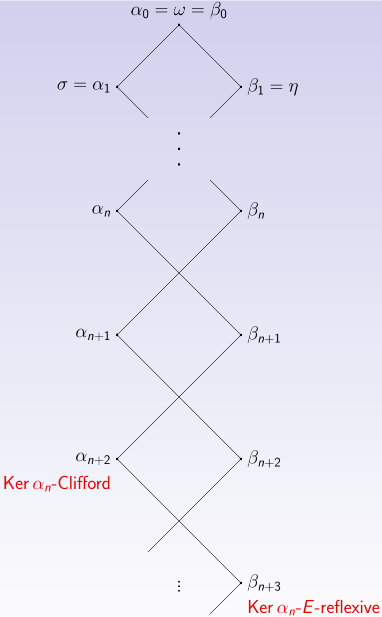
● Fill ● Refine ● Add

Question



$\mathcal{TK}$ min-network of ω

$\mathcal{TK}$-network on inverse semigroup



$\mathcal{TK}$ min-network of ω

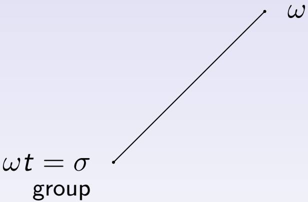
$\mathcal{T}\mathcal{V}$-network of ω

$$\rho\,\mathcal{K}\,\theta \iff \mathsf{Ker}\,\rho = \mathsf{Ker}\,\theta$$

$$\rho\,\mathcal{U}\,\theta \iff \rho\cap\leq\, = \theta\cap\leq$$

$$\mathcal{V} = \mathcal{U}\cap\mathcal{K}$$

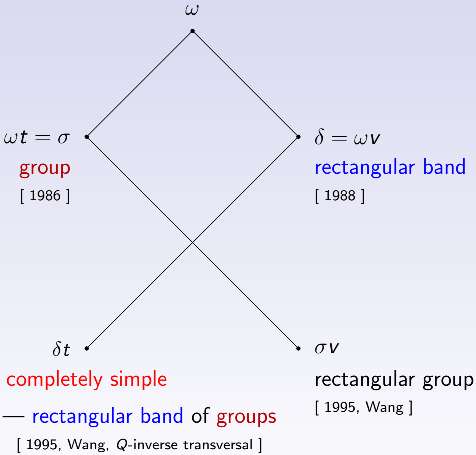
Inverse semigroup $\qquad \mathcal{V} = \varepsilon$



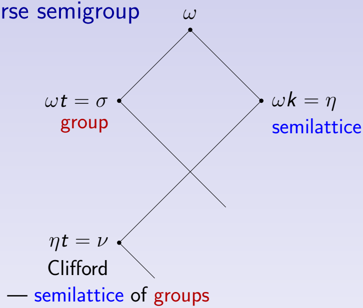
$\mathcal{T}\mathcal{V}$-network of ω

$\mathcal{TV}$-network of ω

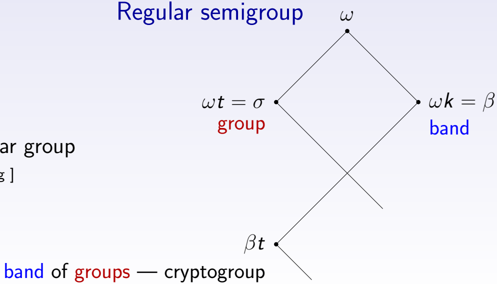
Regular semigroup



Inverse semigroup

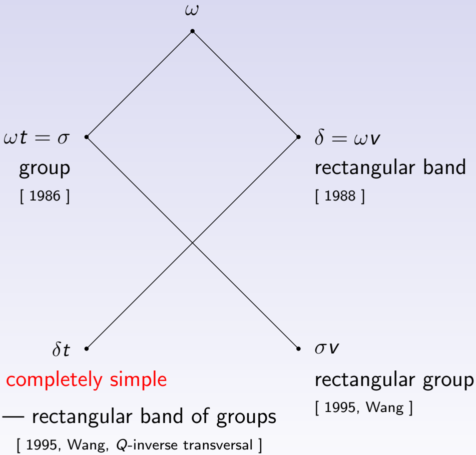


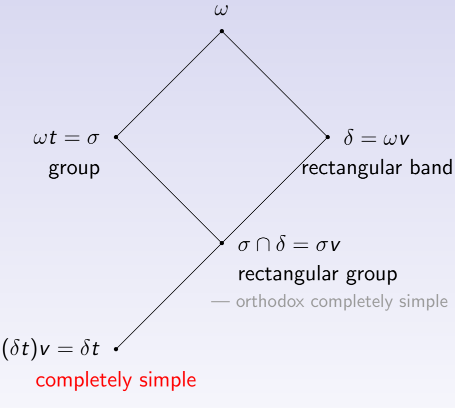
Regular semigroup



$\mathcal{TV}$-network of ω

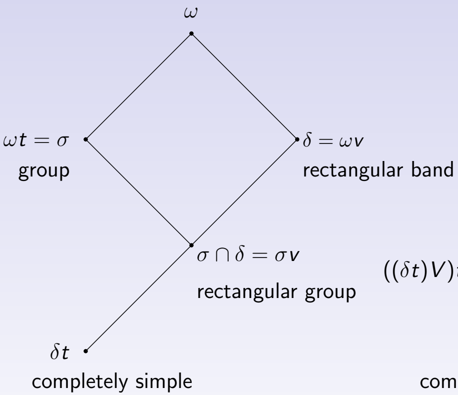
Regular semigroup



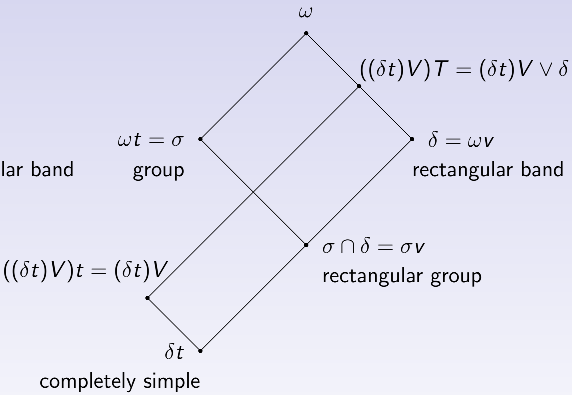


$\mathcal{TV}$-min network of ω

$\mathcal{TV}$-network of ω



$\mathcal{TV}$-min network of ω



$\mathcal{TV}$-network of ω

$\mathcal{TV}$-network of ε

Theorem

For a congruence ρ on a regular semigroup S.

(1) ρt is over bands $\iff \rho t = \rho \cap \tau \Longrightarrow \rho$ is over E-unitary semigroups;

(2) ρt is over rectangular bands $\iff \rho t = \rho \cap \varepsilon V \Longrightarrow \rho$ is over rectangular groups;

(3) ρv is over groups $\iff \rho v = \rho \cap \mu \Longrightarrow \rho$ is over completely simple semigroups;

(4) ρk is over groups $\iff \rho k = \rho \cap \mu \Longrightarrow \rho$ is over cryptogroups.

Corollary

On a regular semigroup S, the following statements hold.

(1) τT is over E-unitary semigroups;

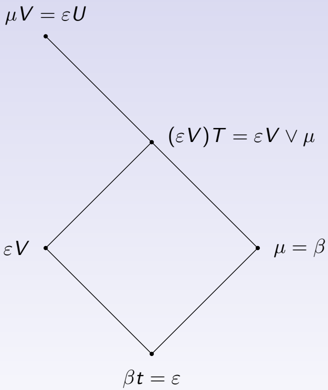
(2) $(\varepsilon V)T$ is over rectangular groups;

(3) μV is over completely simple semigroups;

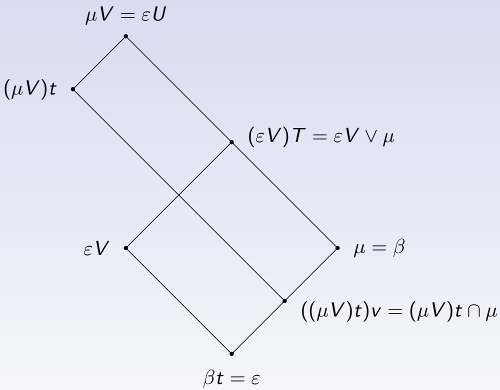
(4) μK is over cryptogroups.

$\mathcal{TV}$-network of ε

Cryptogroup

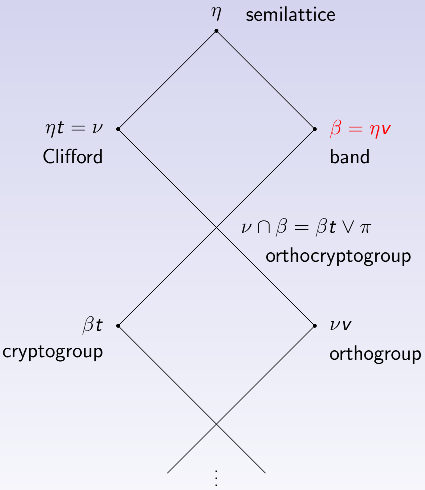


$\mathcal{TV}$-max network of ε



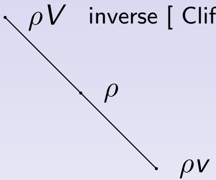
$\mathcal{TV}$-network of ε

$\mathcal{TV}$-network of η



$\mathcal{TV}$-min network of η

$\mathcal{V}$-classes of special congruences



ρV inverse [Clifford, group, semilattice]

ρv orthodox [orthogroup, rectangular group, band]

- orthogroup

  orthodox completely regular semigroup

- rectangular group

  orthodox completely simple semigroup;

  equivalently, a direct product of a rectangular band and a group

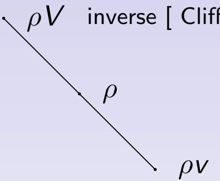
$\mathcal{V}$-classes of special congruences

[1995, Wang]

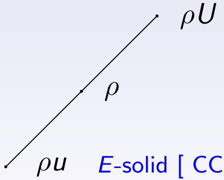
S	orthodox	orthogroup	rectangular group	band
$\Longleftrightarrow$	$\varepsilon V = \gamma$	$\tau V = \nu$	$\varepsilon V = \sigma$	$\varepsilon V = \eta$
$\Longleftrightarrow \quad \forall \rho \in \mathcal{C}(S)$	$\rho V = \rho \vee \gamma$	$\rho V = \rho \vee \nu$	$\rho V = \rho \vee \sigma$	$\rho V = \rho \vee \eta$
$\Longleftrightarrow \quad \rho V$ is	inverse	Clifford	group	semilattice
$\Longleftrightarrow \quad S$ is coextension of	inverse semigroup by rectangular bands	Clifford semigroup by rectangular bands	group by rectangular bands	

$\mathcal{U}$-classes of special congruences

[1995, Wang]



- ρV inverse [Clifford, group, semilattice]
- ρv orthodox [orthogroup, rectangular group, band]



- ρU inverse [Clifford, group]
- ρu E-solid [CCS, CGS]

$\mathcal{U}$-classes of special congruences [1995, Wang]

S	E-solid	CCS	CGS	completely regular
$\Longleftrightarrow \quad \mathscr{U}^0$	inverse	Clifford	group	semilattice
$\Longleftrightarrow \quad \rho U$	inverse	Clifford	group	semilattice

- E-solid
  $\mathcal{R}|_E \circ \mathcal{L}|_E = \mathcal{L}|_E \circ \mathcal{R}|_E$
- CCS
  coextensions of Clifford semigroups by completely simple semigroups
- CGS
  coextensions of groups by completely simple semigroups

Congruence

- kernel-trace approach
- $\mathcal{T}$, $\mathcal{K}$-relation
- congruence networks
- operator semigroup

Operator semigroup

Four operators:

$$T : \lambda \mapsto \lambda T, \quad t : \lambda \mapsto \lambda t, \quad K : \lambda \mapsto \lambda K, \quad k : \lambda \mapsto \lambda k.$$

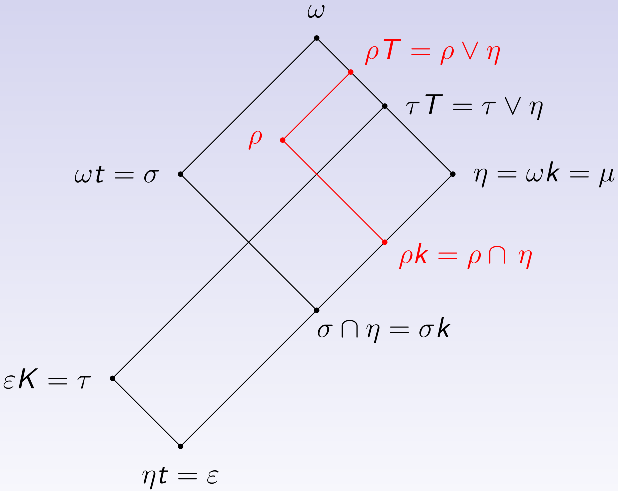
$$\Gamma = \{T, t, K, k\}$$

- $\mathcal{TK}$-network

  ρ, ρT, ρt, ρK, ρk, ρTK, ρTk, $\cdots$

- Γ^+, $\mathcal{TK}$-operator semigroup [1992, Petrich]
  - relation Σ — valid in all networks of congruences
  - $\Gamma^+/\Sigma^\sharp$

$\mathcal{TK}$-operator semigroup for Clifford semigroups



$\mathcal{TK}$-network of ω

Lemma

Operators Γ satisfy the following relations Σ.

(1) $K^2 = kK = K, \qquad k^2 = Kk = k,$

$t^2 = Tt = t, \qquad T^2 = tT = T;$

(2) $KTK = TKT = TK, \quad tkt = ktk = kt;$

(3) $tKt = tK;$

(4) $kT = Tk.$

Denote

$$\varepsilon = kt, \quad \tau = ktK, \quad \tau \vee \eta = ktKT, \quad \eta = kT,$$
$$\omega = TK, \quad \sigma = TKt, \quad \sigma \cap \eta = TKtk.$$

Let

$$\Delta = \{\varepsilon, \sigma, \eta, \tau, \sigma \cap \eta, \tau \vee \eta, \omega\}.$$

Theorem

Let S be a Clifford semigroup. The set

$$\Omega = \{ K, \ KT, \ Kt, \ KtK, \ Ktk, \ KtKT, \ k, \ t, \ tk, \ tK, \ tKT, \ T \} \cup \Delta$$

is a system of representatives for the congruence on Γ^+ generated by the relations Σ.

Theorem

The $\mathcal{TK}$-operator semigroup for Clifford semigroups is $\Gamma^+/\Sigma^\sharp$.

other $\mathcal{TK}$-operator semigroups

- completely simple semigroup [1994, Petrich]
- cryptogroup [2000, Wang]
- bisimple ω-semigroup [2000, Wang]
- E-unitary completely regular semigroup [2001, Luo – Wang]
- free monogenic inverse semigroup [2014, Long – Wang]

Review

- congruence

  $\rho = \rho_{(\text{tr}\,\rho, \text{Ker}\,\rho)}$

- $\mathcal{T}$, $\mathcal{K}$, $\mathcal{U}$, $\mathcal{V}$

  $\rho\,\mathcal{T}\,\theta \iff \text{tr}\,\rho = \text{tr}\,\theta,$

  $\rho\,\mathcal{K}\,\theta \iff \text{Ker}\,\rho = \text{Ker}\,\theta,$

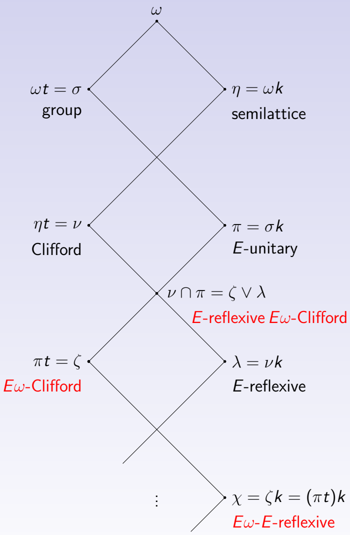
  $\rho\,\mathcal{U}\,\theta \iff \rho \cap \leq = \theta \cap \leq,$

  $\mathcal{V} = \mathcal{U} \cap \mathcal{K}.$

- congruence network

- operator semigroup

  $\Gamma^+/\Sigma^\sharp$, where $\Gamma = \{T, t, K, k\}$.



$\mathcal{T}\mathcal{K}$-min network of ω