CW-decompositions, Leray numbers and the representation theory and cohomology of left regular band algebras

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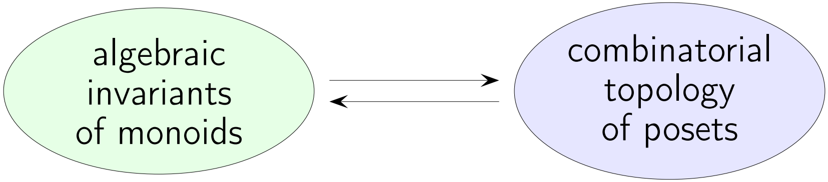
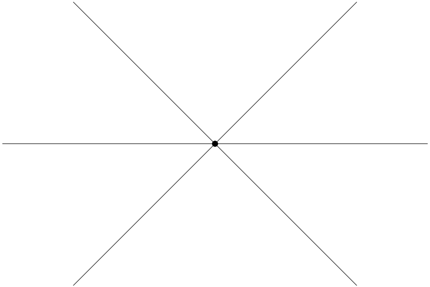

algebraic
invariants
of monoids
combinatorial
topology
of posets

The monoid of faces of a central hyperplane arrangement

a set of hyperplanes partitions $\mathbb{R}^n$ into *faces*:



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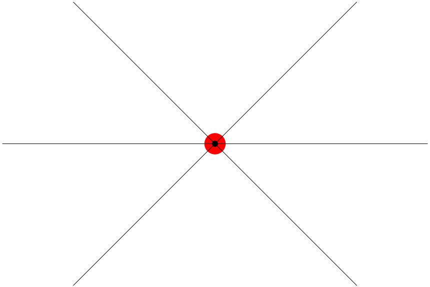

the origin

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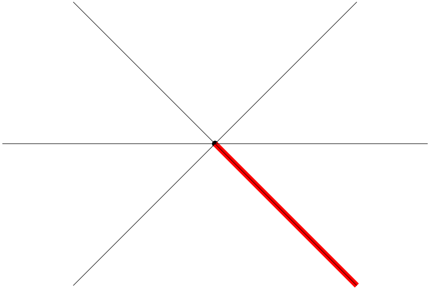

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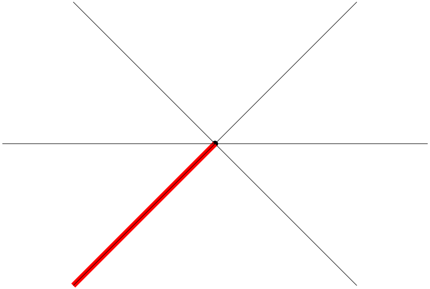

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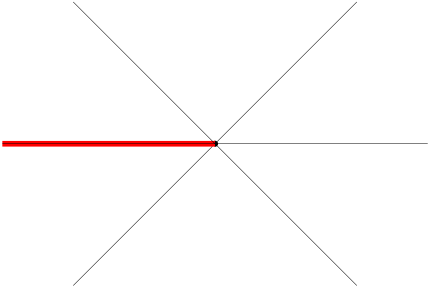

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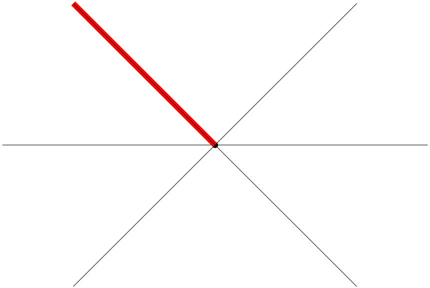

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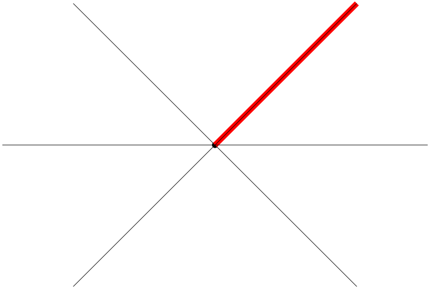

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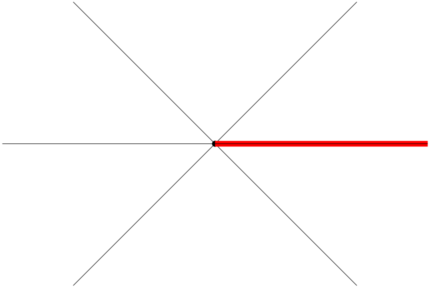

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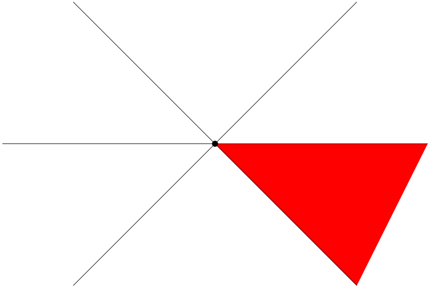

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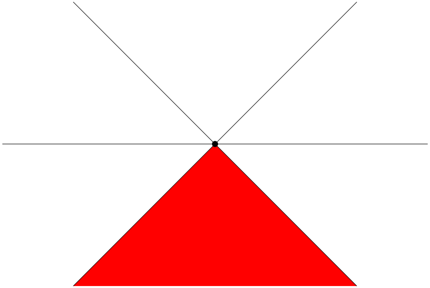

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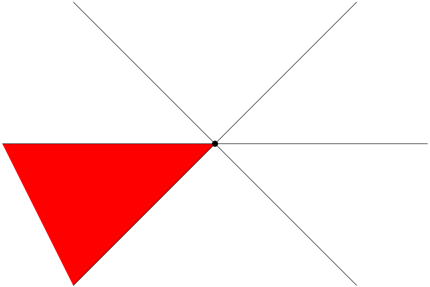

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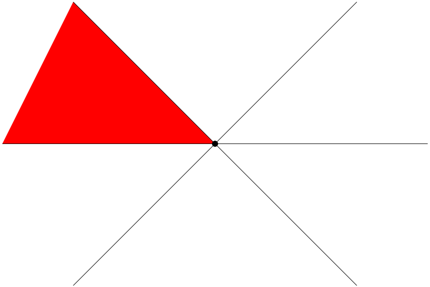

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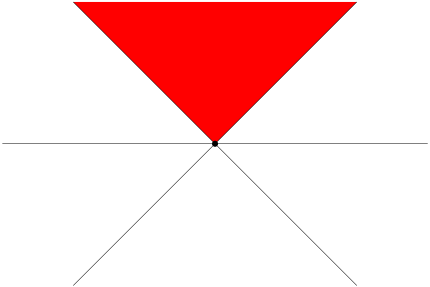

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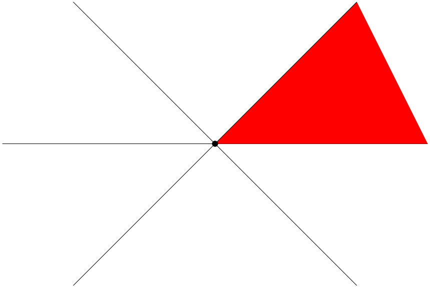

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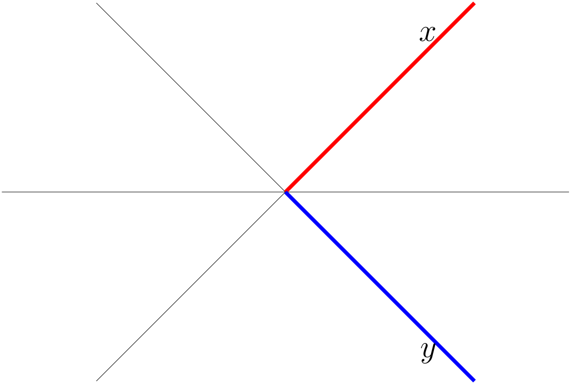
$$xy := \begin{cases} \text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y \end{cases}$$



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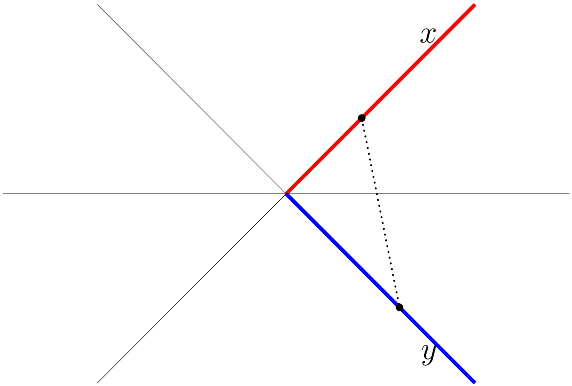

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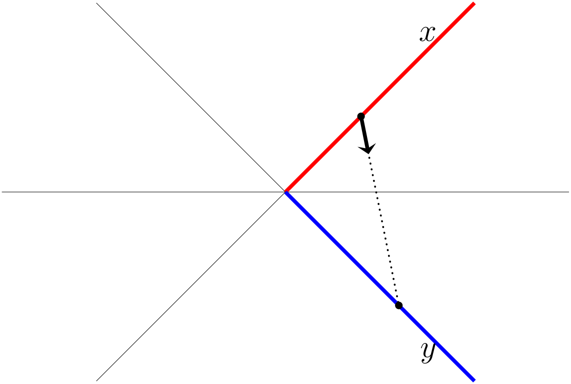

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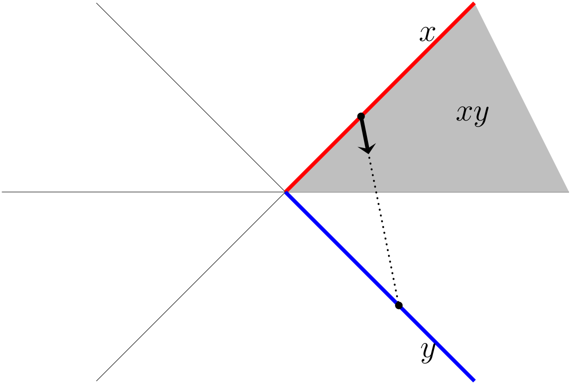

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Remarks

- *Informally: identities say ignore "repetitions".*
- *We consider only finite monoids here.*

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$\Delta((B, \leq))$ is contractible, since 1 is a cone point.

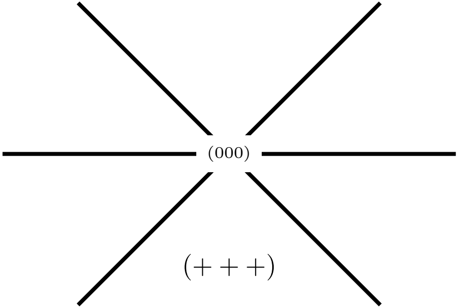
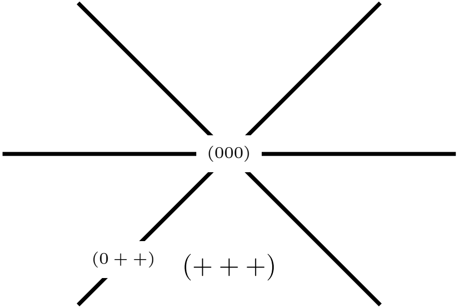


Figure: The sign sequences of the faces of the hyperplane arrangement in $\mathbb{R}^2$ consisting of three distinct lines. The geometric product is just multiplication in $\{0, +, -\}^3$.



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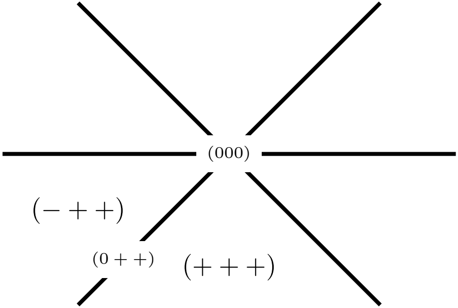

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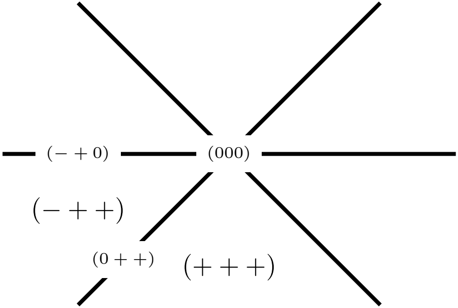

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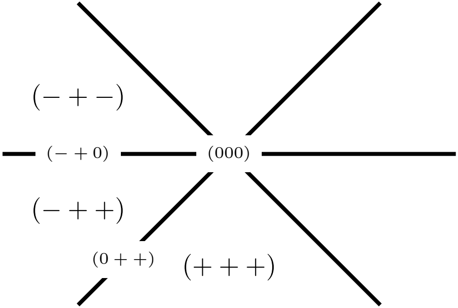

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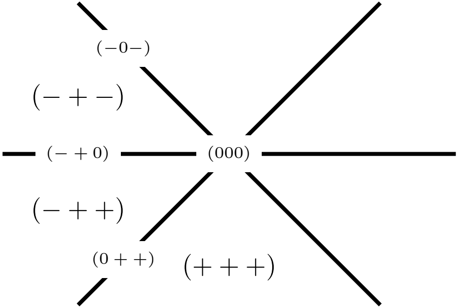

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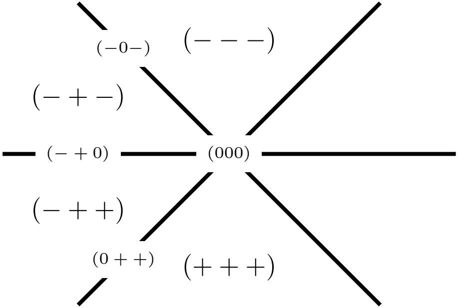

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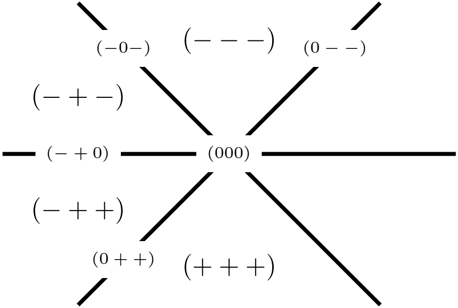

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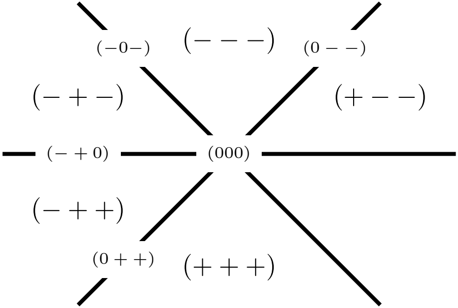

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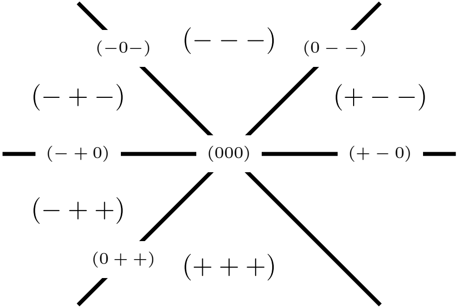

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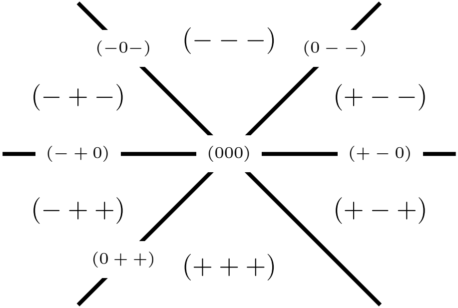

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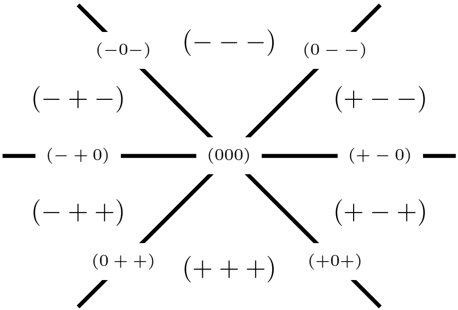

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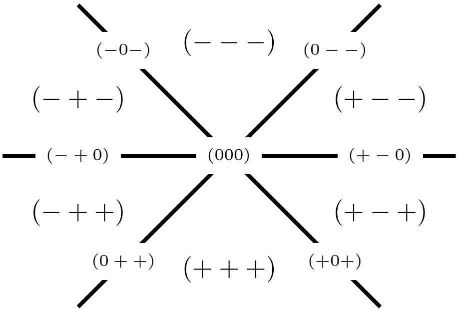

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All hyperplane arrangement LRBs are **submonoids** of $\{0,+,-\}^n$, where $n =$ the number of hyperplanes.

Representation Theory of LRBs

- Simple $\mathbb{K}B$-modules and its Jacobson Radical

Let $\Lambda(B)$ denote the lattice of principal left ideals of B, ordered by inclusion:

$$\Lambda(B) = \{Bb : b \in B\} \qquad\qquad Ba \cap Bb = B(ab)$$

Monoid surjection:

$$\begin{aligned} \sigma : B &\to \Lambda(B) \\ b &\mapsto Bb \end{aligned}$$

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where $\overline{\sigma} : \mathbb{K}B \to \mathbb{K}(\Lambda(B))$ is the extended morphism. $\mathbb{K}(\Lambda(B))$ is semisimple and so simple $\mathbb{K}B$-modules S_X are indexed by $X \in \Lambda(B)$.

Semisimple Quotient and Simple Modules

$$\mathbb{K}B/\operatorname{rad}(\mathbb{K}B) \cong \mathbb{K}B/\ker(\overline{\sigma}) \cong \mathbb{K}\Lambda(B) \cong \mathbb{K}^{\Lambda(B)}$$

For each $X \in \Lambda(B)$, the corresponding simple module is 1 dimensional and is given by the following action.

$$\rho_X(a) = \begin{cases} 1, & \text{if } \sigma(a) \geq X, \\ 0, & \text{otherwise} \end{cases}$$

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We see then that $\mathbb{K}B$ is a basic algebra: All of its simple modules are 1 dimensional. Equivalently, $\mathbb{K}B$ has a faithful representation by triangular matrices.

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Tsetlin Library : "use a book, then put it at the front"

Free Partially-Commutative LRB

The *free partially-commutative LRB* $F(G)$ on a graph $G = (V, E)$ is the LRB with presentation:

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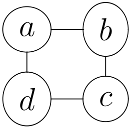
- $F(K_n) =$ free commutative LRB, that is the free semilattice, on n generators.
- LRB-version of the Cartier-Foata *free partially-commutative monoid* (aka *trace monoids*).

Acyclic orientations

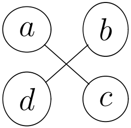
Elements of $F(G)$ correspond to acyclic orientations of induced subgraphs of the complement $\overline{G}$.

Example

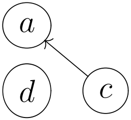
$G =$



$\overline{G} =$



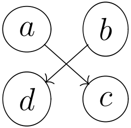
Acyclic orientation on induced subgraph on vertices $\{a, d, c\}$:



In $F(G)$: $cad = cda = dca$ (c comes before a since $c \to a$)

Random walk on $F(G)$

States: acyclic orientations of the complement $\overline{G}$



Step: left-multiplication by a generator (vertex) reorients all the edges incident to the vertex away from it

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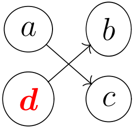

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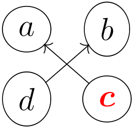

Step: left-multiplication by a generator (vertex) reorients all the edges incident to the vertex away from it

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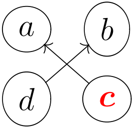

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Athanasiadis-Diaconis (2010): studied this chain using a different LRB (graphical arrangement of G)

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- It is an LRB whose $\mathcal{R}$ order has Hasse diagram a tree and $\mathcal{L}$ order is the Hasse diagram of Λ.

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- oriented matroids

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LRBs are everywhere :

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Other combinatorial semigroups :

Ayyer, Denton, Hivert, Schilling, Steinberg, Thiery, . . .

Goal : Extensions

$$\mathrm{Ext}^n_B(S, T)$$

for simple modules S and T

Question : Given two modules S and T, *how* can they be combined to make new modules M ?

$$\boxed{S \subseteq M \quad \text{and} \quad T \cong M/S}$$

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$\mathrm{Ext}^1(S,T)$: vector space of equiv. classes of SES

Main theorem as a haiku

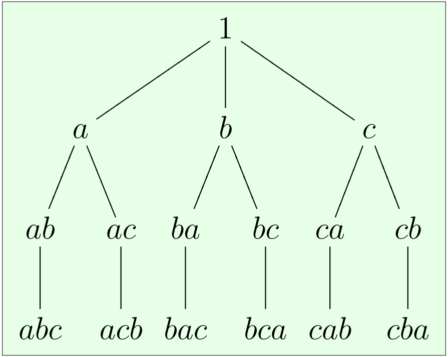
For a LRB

the Extensions are poset

cohomology.

$$(B, \leq\ )$$

$$x \leq y \Leftrightarrow yx = x$$



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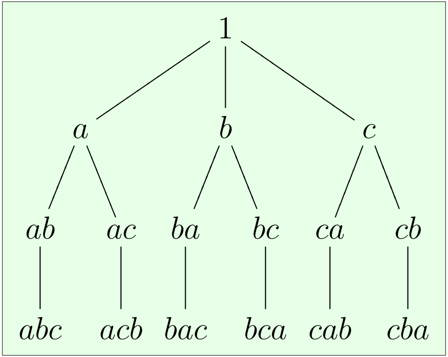
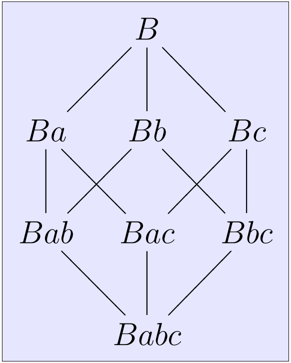
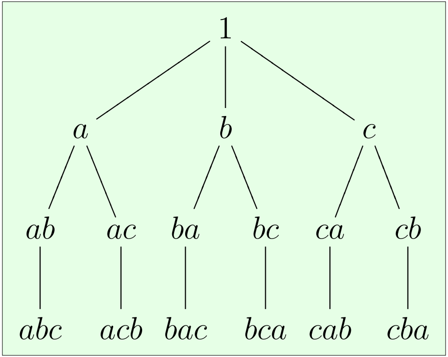


hyperplane arrangements :
face relation

$$\big(\Lambda(B), \subseteq\big) \qquad\qquad \big(B, \leq\big)$$

$$Bx = \{bx : b \in B\} \qquad\qquad x \leq y \Leftrightarrow yx = x$$





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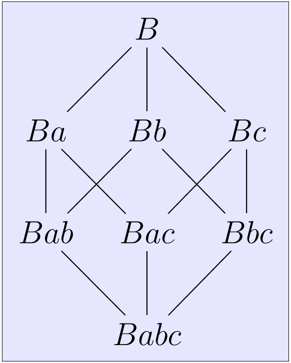

hyperplane arrangements :
intersection lattice

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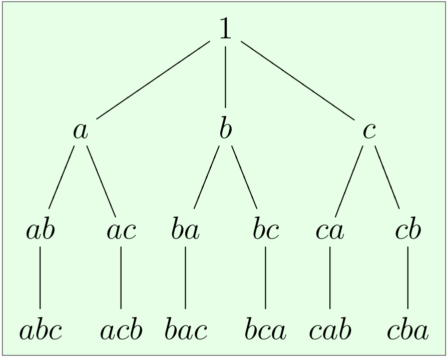

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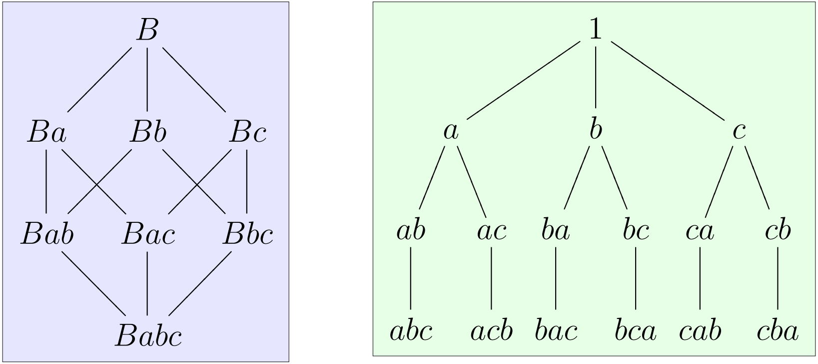

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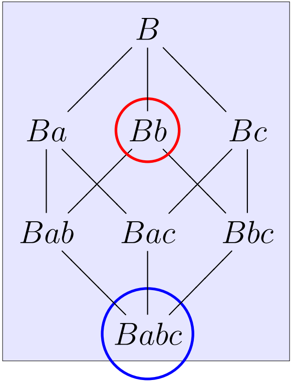

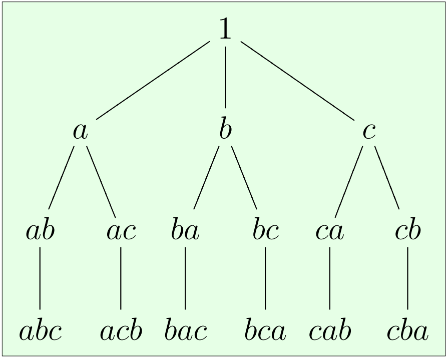

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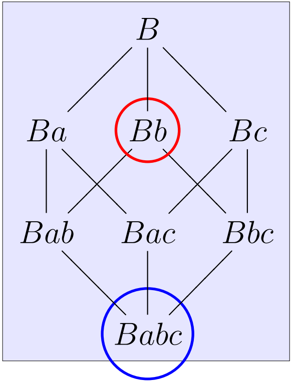

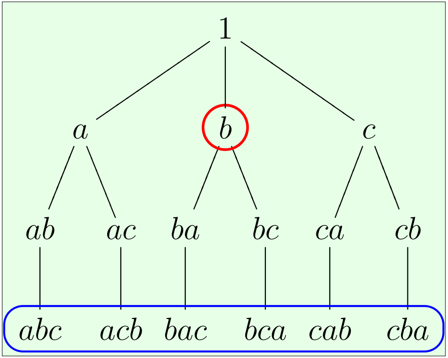

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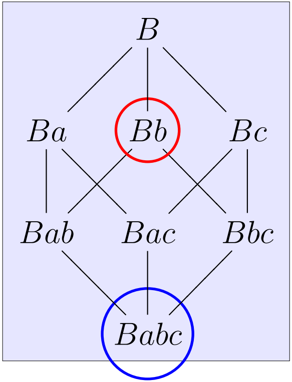

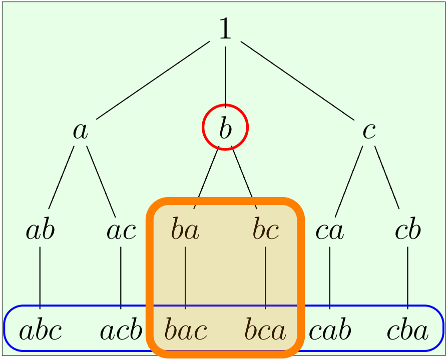

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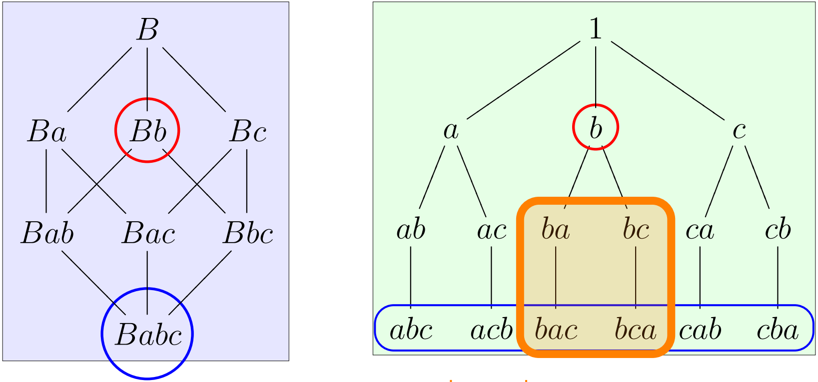
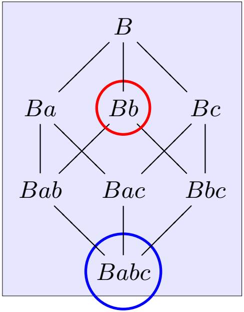
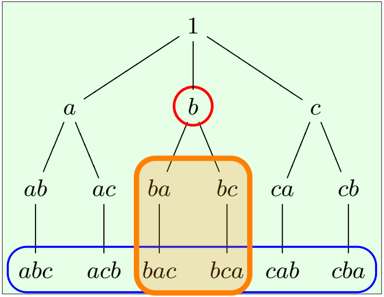


hyperplane arrangements :
restriction and contraction

Main Theorem (M-S-S)

$$\mathrm{Ext}^n(S_X, S_Y) \cong \widetilde{H}^{n-1}(\Delta B_{[X,Y)})$$





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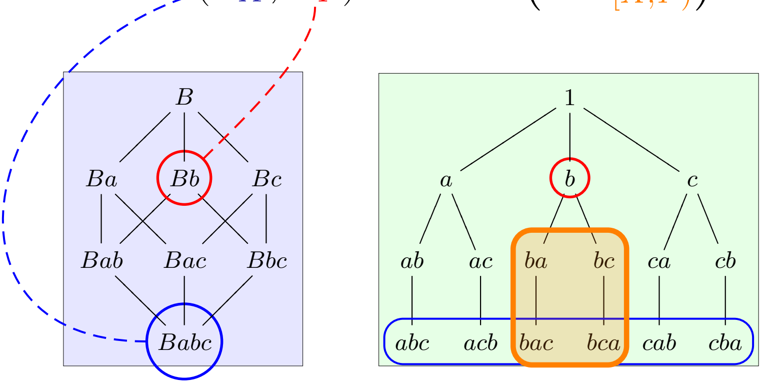

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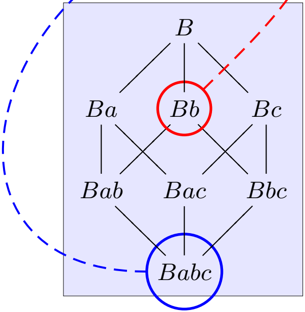

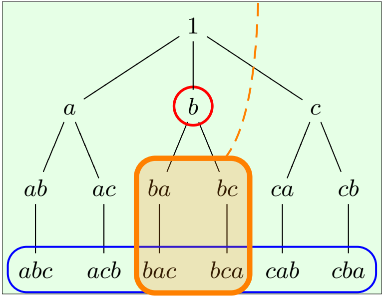

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- $\Delta B_{[X,Y)}$ is the *order complex* of $B_{[X,Y)}$

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B

Ba Bb Bc

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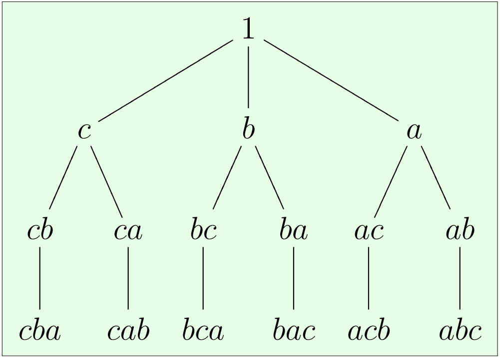
$Babc$



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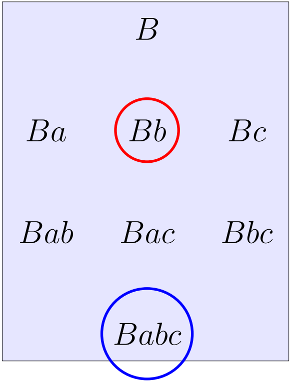

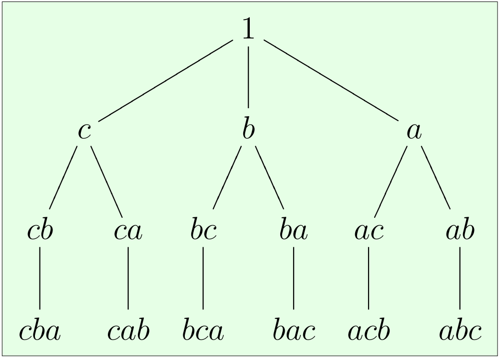

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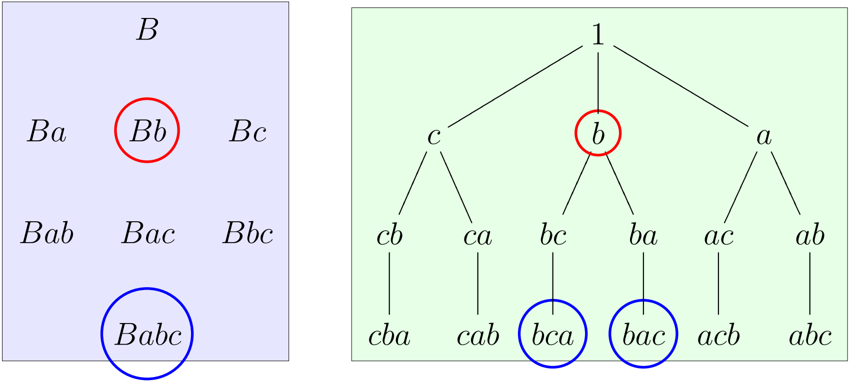

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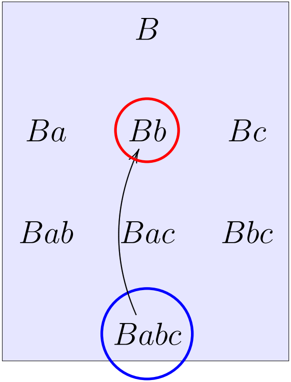

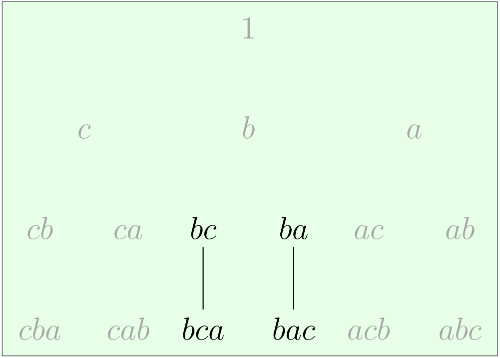

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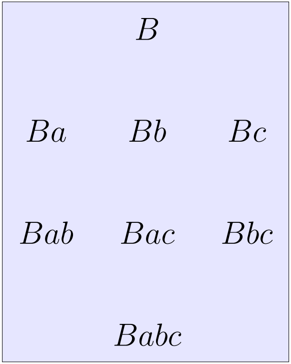

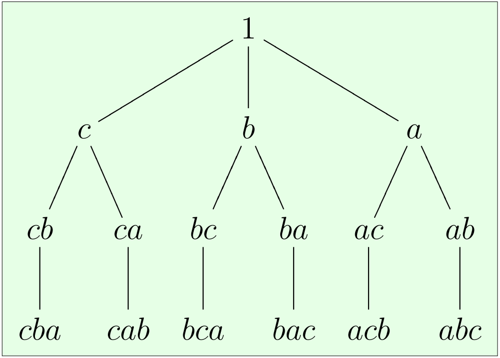

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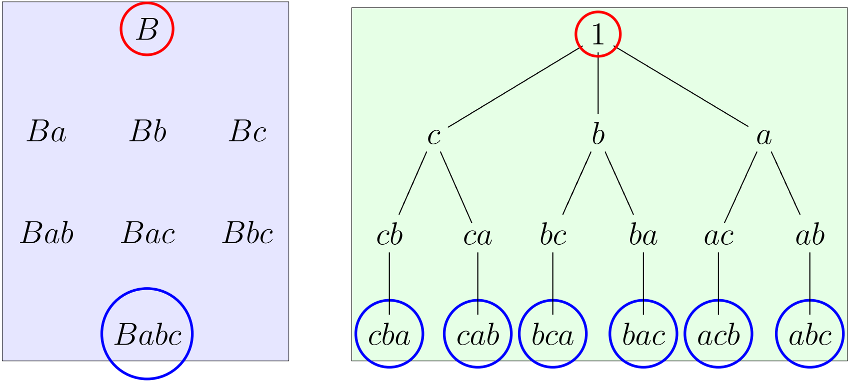

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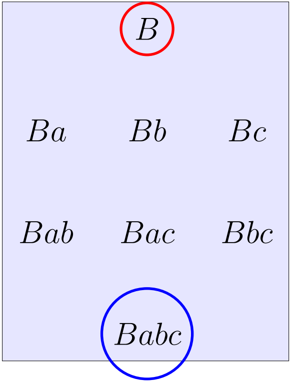

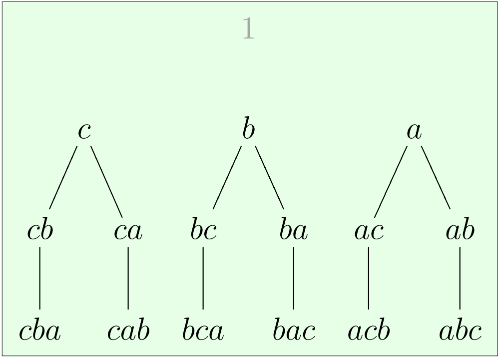

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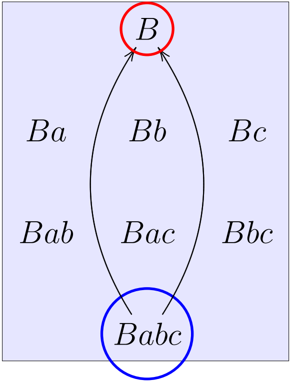

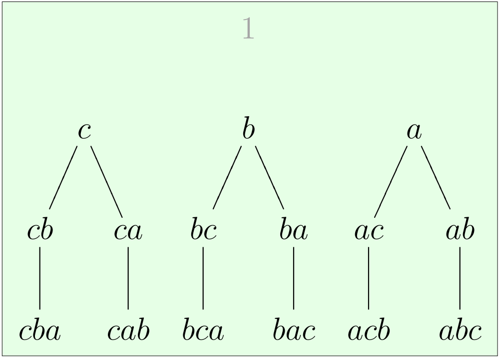

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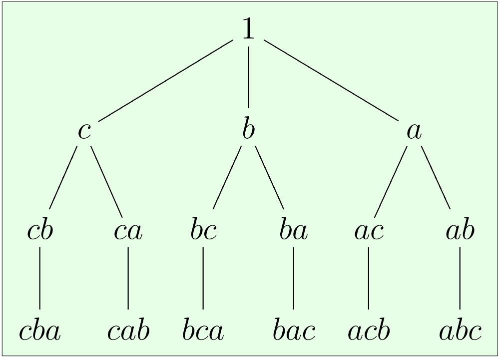

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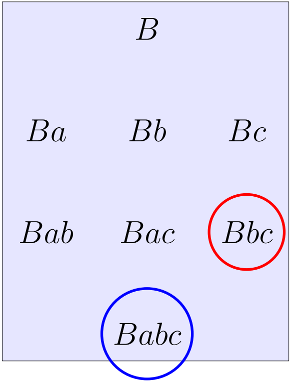

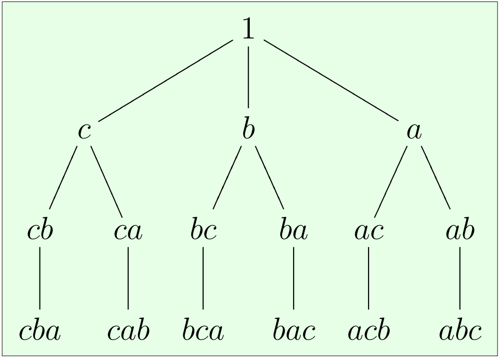

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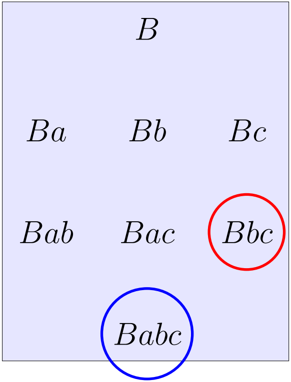

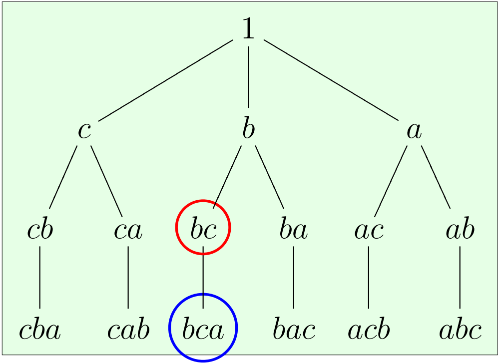

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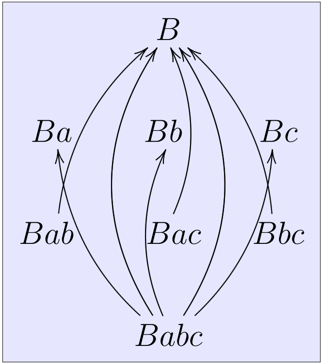
$$\begin{array}{cccccc} & & & 1 & & \\ c & & & b & & a \\ cb & ca & bc & ba & ac & ab \\ cba & cab & bca & bac & acb & abc \end{array}$$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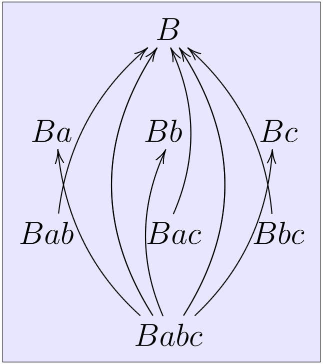

Quiver of an algebra is the directed graph where

- *vertices* are the simple modules
- $\#$ *arrows* $S \to T$ is $\dim \mathrm{Ext}^1(S,T)$

Global dimension

Let A be a finite dimensional algebra.

- The projective dimension of an A-module M is the minimum length of a projective resolution

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

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- For finite-dimensional algebras, the sup can be taken over simple modules.

Global dimension and Leray numbers

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5. $\mathbb{K}F(G)$ is hereditary iff G is chordal, that is, has no induced cycles greater than length 3.

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CAT(0) cube complexes :

- $\Lambda(B)$ is Cohen-Macaulay (we prove the incidence algebra is Koszul)

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This is used to compute all the spaces $\mathrm{Ext}^n(S, T)$ between simple $K(B)$ modules, S, T when K is a field and obtain the main theorem.

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Theorem
$(P, \leq)$ is a CW poset if and only if $(P, \leq)$ is graded and for every $p \in P$, $\{q | q < p\}$ is isomorphic to a sphere of dimension $rank(p) - 1$.

Definition
An LRB B is a CW LRB if every poset $(B_X, \leq)$, $X \in \Lambda(B)$ is a CW poset.

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The following are examples of CW LRBs.

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(d) *KB is a Koszul algebra and its Koszul dual is isomorphic to the dual of the incidence algebra of $\Lambda(B)$.*

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(f) Every open interval of $\Lambda(B)$ is a Cohen-Macauley poset.

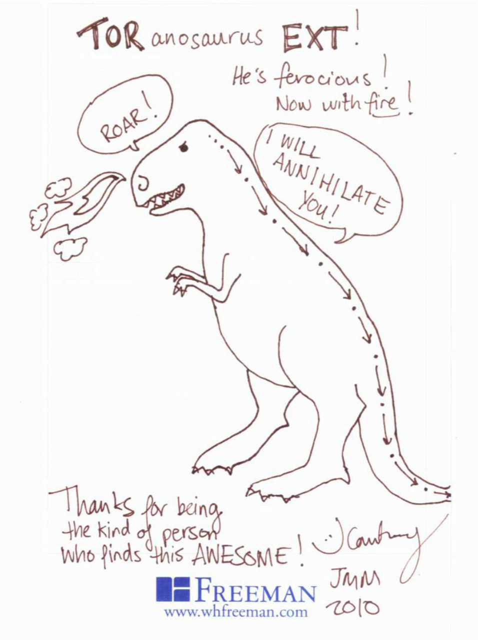

image from Sean Sather-Wagstaff