On free products and amalgams of pomonoids

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Joint work with Bana Al Subaiei

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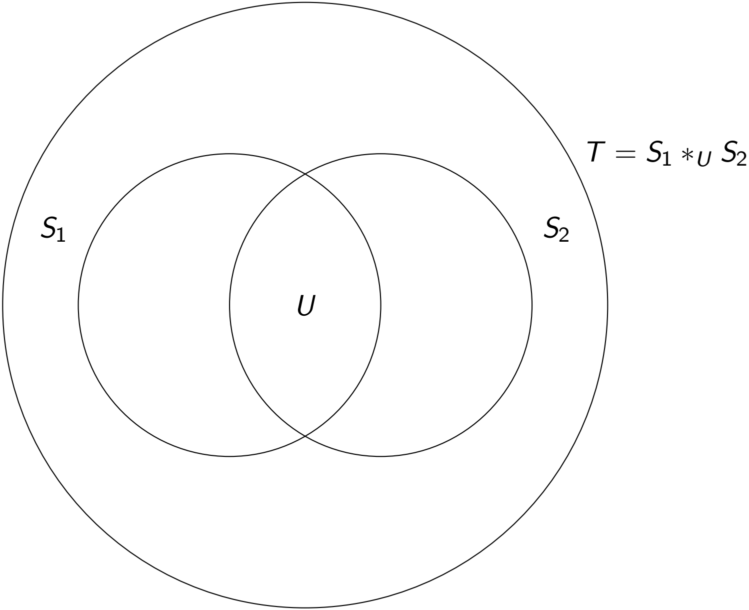
Outline

- Howie 1962 Semigroup amalgams (unitary properties), Cohn 1959 Ring amalgams (flatness properties)
- Me 1985 - Ring approach to monoid amalgams
- Amalgamation in the category of partially ordered monoids - Fakhruddin in the 1980s.
  In the category of commutative pomonoids, every absolutely flat commutative pomonoid is a weak amalgamation base and every commutative pogroup is a strong amalgamation base.
- Bulman-Fleming and Sohail in 2011. Pogroups are poamalgmation bases in the category of pomonoids.

In this talk I will look at recent joint work with Bana Al Subaiei generalising some earlier results on amalgams of monoids and extension properties of acts over monoids to pomonoids and in particular on ordered version of *unitary* properties which generalise Howie's original work on semigroup and monoid amalgams.


$T = S_1 *_U S_2$
S_1
S_2
U

If $f : U \to S$ is a monoid morphism then we can view S as an *act over* U with multiplication given by $s \cdot u = sf(u)$.

$A_U, {}_UB$
the tensor product $A \otimes_U B$ is the quotient of $A \times B$ such that

$$(au, b) \equiv (a, ub)$$

$U \subseteq S, X_S, Y_U, f : X \to Y$
the free $S-$*extension of* X *and* Y, $F(S; X, Y)$ is the quotient of $Y \otimes_U S$

$$f(x) \otimes s \equiv f(xs) \otimes 1$$

The free $S-$extension of X and Y, $F(S; X, Y)$, can also be constructed as the pushout within the category of $S-$acts of the diagram

$$\begin{array}{ccc} X \otimes_U S & \longrightarrow & Y \otimes_U S \\ \downarrow & & \\ X & & \end{array}$$

$$
\begin{array}{ccccccc}
U & \longrightarrow & S_2 & & & & \\
\downarrow & & \downarrow & & & & \\
S_1 = W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & \cdots
\end{array}
$$

Where $W_2 = W_1 \otimes_U S_2$
$W_3 = F(S_1; W_1, W_2)$
$W_4 = F(S_2; W_2, W_3)$ etc

$$W_3 : s_1 \otimes 1 \otimes s_1' \equiv s_1 s_1' \otimes 1 \otimes 1 \text{ in } S_1 \otimes_U S_2 \otimes_U S_1$$

Theorem

U is an amalgamation base *if and only if U has the* extension property *in each overmonoid S.*

U has the extension property in S if for all $X_U, {}_UY$

$$X \otimes_U U \otimes_U Y \to X \otimes_U S \otimes_U Y$$

is an embedding

$U \subseteq S$ is (right) unitary in S if $su \in U \Rightarrow s \in U$

If we think of S as a right $U-$act then this is equivalent to saying that U is a direct summand of S, i.e. $S \cong U \dot{\cup} V$

($A \subseteq B$: A is *unitary in* B if $bu \in A \Rightarrow b \in A$)

$$X \otimes_U S \otimes_U Y \cong (X \otimes_U U \otimes_U Y) \dot{\cup} (X \otimes_U V \otimes_U Y)$$

A monoid S is said to be a *partially ordered monoid* or a *pomonoid* if S is endowed with a partial order $\leq$ which is compatible with the binary operation on S in the following manner

$$\forall s, t, u \in S, t \leq u \Rightarrow st \leq su \text{ and } ts \leq us.$$

A map $f\colon X \to Y$, where X and Y are posets, is said to be *monotone* if for all $x, y \in X, x \leq y \Rightarrow f(x) \leq f(y)$ and pomonoids together with monotone homomorphisms form a category.

If S is a pomonoid and A is a non empty poset, then A is called a *right $S-$poset* if A is a right $S-$act and the action is monotonic in each of the variables. That is to say

- $a1 = a$ and $a(st) = (as)t$ for all $s, t \in S, a \in A$;
- if $a \leq b \in A, s \in S$ then $as \leq bs$;
- if $a \in A, s \leq t \in S$ then $as \leq at$.

If A and B are $S-$posets then the map $f : A \rightarrow B$ is said to be an *$S-$poset morphism* when f is both monotonic and a morphism of $S-$acts.

Theorem

Let $(U; S)$ be a weak poamalgamation base in the category of pomonoids. Then U has the poextension property in every containing pomonoid S.

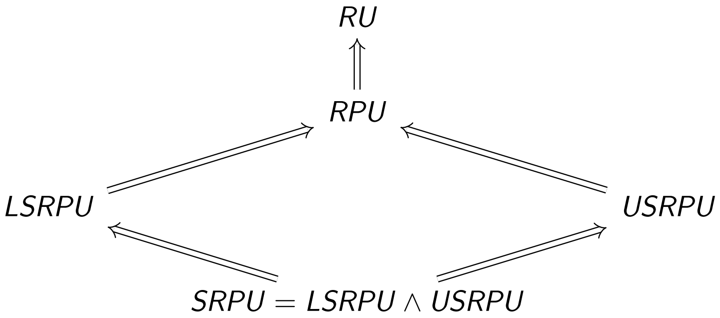
Let U be a subpomonoid of the pomonoid S and let $v, u, u_1, u_1', \ldots u_n, u_n' \in U$, $s, s_1, s_2, \ldots s_n \in S$. We shall say that

1. U *is upper strongly right pounitary in* S (USRPU) if $v \leq su \Rightarrow s \in U$;
2. U *is lower strongly right pounitary in* S (LSRPU) if $su \leq v \Rightarrow s \in U$;
3. U *is strongly right pounitary in* S (SRPU) if $(v \leq su \text{ or } su \leq v) \Rightarrow s \in U$;
4. U *is right pounitary in* S (RPU) if whenever there exists $n \geq 1$ such that

$$u \leq s_1 u_1, s_1 u_1' \leq s_2 u_2, \ldots s_n u_n' \leq v$$

then $s_1, s_2, \ldots, s_n \in U$;

5. U *is right unitary in* S (RU) if $su = v \Rightarrow s \in U$.


RU
RPU
LSRPU
USRPU
SRPU = LSRPU ∧ USRPU

Theorem

Let U be a (left, right) pounitary subpomonoid of a pomonoid S. Then U has the (right, left) poextension property in S.

Notice that if U is strongly right pounitary in S then $S \setminus U$ is a right $U-$poset and S is the coproduct in the category of right $U-$posets of U and $S \setminus U$.

In other words, within the category of right $U-$posets, U is a direct summand of S.

Theorem

Let $[U; S_1, S_2]$ be a pomonoid amalgam. If U is strongly pounitary in both S_1 and S_2 then the amalgam is strongly poembeddable and U has the poextension property in $S_1 *_U S_2$.

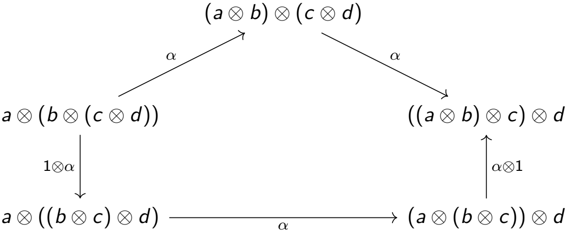
Theorem

Let U be a strongly pounitary subpomonoid of a pomonoid S. Then for every $(U,S)-$poset X and every $(U,U)-$poset Y and every strongly pounitary order embedding $f : X \to Y$ the induced map $g : Y \to F(S;X,Y)$ is a $(U,U)-$strongly pounitary order embedding.

$$
\begin{array}{ccccccc}
U & \longrightarrow & S_2 & & & & \\
\downarrow & & \downarrow & & & & \\
S_1 = W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & \cdots
\end{array}
$$

A *monoidal category* $(\mathcal{V}, \otimes, 1, \alpha, \lambda, \rho)$: category $\mathcal{V}$, bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, object $1 \in \mathcal{V}$ and natural isomorphisms α, λ and ρ, where

1. $\alpha = \alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ is natural for all $a, b, c \in \mathcal{V}$ and where
2. the diagram



   commutes

3. $\lambda = \lambda_a : 1 \otimes a \cong a \qquad \rho = \rho_a : a \otimes 1 \cong a$ are natural for all $a \in \mathcal{V}$.

4. the diagram

$$\begin{array}{ccc} a \otimes (1 \otimes b) & \xrightarrow{\alpha} & (a \otimes 1) \otimes b \\ & \searrow^{1 \otimes \lambda} & \downarrow{\scriptstyle \rho \otimes 1} \\ & & a \otimes b \end{array}$$

   commutes

5. $\lambda_1 = \rho_1 : 1 \otimes 1 \cong 1.$

It is not too difficult to check that the following are all examples of monoidal categories :

- $(Set, \times, 1)$
- $(Abg, \otimes_{\mathbb{Z}}, \mathbb{Z})$,
- $(Top, \times, 1)$,
- $(Cat, \times, 1)$,
- $((\mathbb{R}^+, \leq), +, 0)$.

A *monoid* in a monoidal category $(\mathcal{V}, \otimes, 1, \alpha, \lambda, \rho)$, consists of an object $m \in \mathcal{V}$, together with maps $\mu : m \otimes m \to m, \epsilon : 1 \to m$ such that

$$\begin{array}{ccccc} m \otimes (m \otimes m) & \xrightarrow{\alpha} & (m \otimes m) \otimes m & \xrightarrow{\mu \otimes 1} & m \otimes m \\ \downarrow{\scriptstyle 1 \otimes \mu} & & & & \downarrow{\scriptstyle \mu} \\ m \otimes m & & \xrightarrow[\mu]{\hspace{6em}} & & m \end{array}$$

$$\begin{array}{ccccc} 1 \otimes m & \xrightarrow{\epsilon \otimes 1} & m \otimes m & \xleftarrow{1 \otimes \epsilon} & m \otimes 1 \\ & {\scriptstyle \lambda} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \rho} & \\ & & m & & \end{array}$$

A *triple* (or monad) (T, μ, ϵ) in a category $\mathcal{V}$ consists of a functor $T : \mathcal{V} \to \mathcal{V}$ and two natural transformations $\epsilon : 1_{\mathcal{V}} \dot{\to} T, \mu : T^2 \dot{\to} T$ such that

$$
\begin{array}{ccc}
T^3 & \xrightarrow{T\mu} & T^2 \\
{\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\
T^2 & \xrightarrow[\mu]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccccc}
T & \xrightarrow{\epsilon T} & T^2 & \xleftarrow{T\epsilon} & T \\
 & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle =} & \\
 & & T & &
\end{array}
$$

commutes.

If (T, μ, ϵ) is a triple in $\mathcal{V}$, then a T–algebra consists of $X \in \mathcal{V}$ and a map $h : TX \to X$ such that

$$\begin{array}{ccc} T^2X & \xrightarrow{Th} & TX \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle h} \\ TX & \xrightarrow[h]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\epsilon_X} & TX \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle h} \\ & & X \end{array}$$

commute.

Let $(\mathcal{V}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category and let (m, μ, ϵ) be a monoid.

Let L_m be the functor given by $L_m = m \otimes —$

natural transformations
$\mu' : L_m^2 \dot{\to} L_m, \mu'_X = (\mu \otimes 1) \circ \alpha_{m,m,x}$

$\epsilon' : 1 \dot{\to} L_m, \epsilon'_X = (\epsilon \otimes 1) \circ (\lambda_x^{-1})$.

Then it is straightforward to show that (L_m, μ', ϵ') is a triple.

The algebras over this triple are the (left) acts.

If $\mathcal{V}$ is the category of SETS then the L_m-acts are the usual (left) acts over a monoid m.

If $\mathcal{V}$ is the category of ABELIAN GROUPS then the L_m-acts are the (left) modules over the ring m.

If $\mathcal{V}$ is the category of POSETS then the L_m-acts are the (left) $m-$posets over the pomonoid m.

Let $(X, f) \in ACT\text{–}m, (Y, g) \in m\text{–}ACT$ and consider the diagram

$$X \otimes (m \otimes Y) \underset{(f \otimes 1_Y) \circ \alpha_{X,m,Y}}{\overset{1_X \otimes g}{\rightrightarrows}} X \otimes Y$$

The coequaliser, in $\mathcal{V}$, (if it exists) of this diagram is called the *tensor product of* X *and* Y *over* m and is written $X \otimes_m Y$.

Theorem

If $\mathcal{V}$ is a cocomplete category and T a triple on $\mathcal{V}$ such that T preserves colimits in $\mathcal{V}$ then $T-$alg is cocomplete.

Theorem

Let $(\mathcal{V}, \otimes, 1, \alpha, \lambda, \rho)$ be a monoidal category and let (m, μ, ϵ) be a monoid in $\mathcal{V}$. If $X\otimes —$ and $X \otimes m\otimes —$ preserve colimits in $\mathcal{V}$ then so does $X\otimes_m —$.

Theorem

- $X \otimes_m m \cong X$,
- $\otimes_1 \cong \otimes$,
- $\alpha : X \to Y, \beta : A \to B, \exists \alpha \otimes \beta : X \otimes_m A \to Y \otimes_m B$.

Theorem

Let $(\mathcal{V}, \otimes, 1, \alpha, \lambda, \rho)$ *be a cocomplete monoidal category and suppose that* $X \otimes —$ *and* $— \otimes X$ *are colimit preserving in* $\mathcal{V}$ *then for all* $X \in \mathcal{V}$*. Then*

- $X \otimes_m m \in$ *Act-m,*
- $(X \otimes_m Y) \otimes_m Z \cong X \otimes_m (Y \otimes_m Z)$ *for all* $X, Y, Z \in \mathcal{V}$*,*
- *the category of monoids in* $\mathcal{V}$ *is cocomplete.*

Let $(\mathcal{V}, \otimes, 1)$ be a monoidal category and consider the monoidal category of endofunctors on $\mathcal{V}$

$$(\mathcal{V}^{\mathcal{V}}, \circ, 1).$$

There is a natural 'embedding' of $\mathcal{V}$ in $\mathcal{V}^{\mathcal{V}}$, $X \mapsto X \otimes —$.

Monoid $m \mapsto m \otimes —$.

The monoids in $(\mathcal{V}^{\mathcal{V}}, \circ, 1)$ are in fact the triples over $\mathcal{V}$.

Theorem

If $\mathcal{V}$ is cocomplete then the category of colimit preserving triples on $\mathcal{V}$ is the category of monoids in the monoidal category of colimit preserving endofunctors over $\mathcal{V}$.