Computation and conjugacy in hypoplactic and sylvester monoids, and other homogeneous monoids

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Joint work with Robert D. Gray & António Malheiro

Young tableaux & the plactic monoid

Let $n \in \mathbb{N}$ and let $A = \{1 < 2 < 3 < \ldots < n\}$.

6					
5	5	7			
3	4	6			
1	1	2	3	5	7

- Rows non-decreasing left to right
- Columns decreasing top to bottom
- Left-justified, shorter rows on top

Schensted's algorithm computes a tableau $P(u)$ from a word $u \in A^*$. Define

$$u \equiv v \iff P(u) = P(v).$$

Theorem (Knuth 1970)

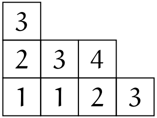
The relation $\equiv$ is a congruence on A^*.

The factor monoid $P_n = A^*/\equiv$ is the Plactic monoid of rank n

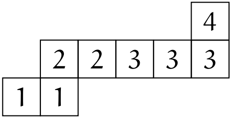
- Connected with combinatorics, quantum groups, symmetric functions, representations of $\mathfrak{sl}_n$ and $\mathfrak{S}_n$.

‘Plactic-like’ monoids

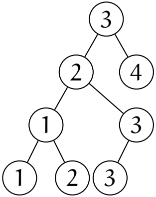
Plactic monoid
Young tableaux



Hypoplactic monoid
Quasi-ribbon tableaux



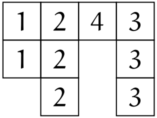
Sylvester monoid
Binary search trees



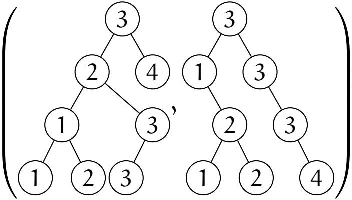
Bell monoid
Set partitions

$$\{\{5,1\},\ \{8,6,4,2\},\ \{7,3\},\ \{9\}\}$$

Stalactite monoid
Stalactite tableaux



Baxter monoid
Pairs of binary search trees



Rewriting systems

A monoid is FCRS if it admits a presentation via a finite complete rewriting system (on some generating set).

- Having a finite complete rewriting system presentation is dependent on the choice of generators.
- Finite derivation type (FDT) is a consequence of FCRS but is not dependent on the choice of generators.

Automaticity & biautomaticity

Let M be a monoid, A a generating set for M, and L a regular language over A such that L maps onto M. Define relations

$$L_a = \{(u, v) \in L \times L : ua =_M v\},$$
$$_aL = \{(u, v) \in L \times L : au =_M v\}.$$

The pair (A, L) is

- a automatic structure for M if L_a is recognizable by a synchronous two-tape automaton for all $a \in A \cup \{\varepsilon\}$;
- an biautomatic structure for M if L_a and $_aL$ are recognizable by synchronous two-tape automata for all $a \in A \cup \{\varepsilon\}$.

A monoid is

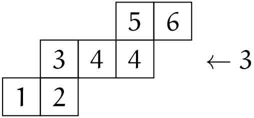
- automatic (AUTO) if it admits an automatic structure;
- biautomatic (BIAUTO) if it admits an biautomatic structure.

Theorem (C, Gray, Malheiro 2015)

P_n is FCRS and BIAUTO.

Quasi-ribbon tableaux

Quasi-ribbon tableau (QRT):



To insert a symbol x into a quasi-ribbon tableau T:

- Break the tableau two parts: $T_{\leqslant}$ is up to and including the top-right-most symbol r such that $r \leqslant x$; the remainder is $T_{>}$.
- Add x to the right of r.
- Attach $T_{>}$ to the top of x.

For a word $w = w_1 w_2 \cdots w_n$.

- Start with an empty QRT insert w_1, then w_2, ..., finally w_n.
- Call the resulting quasi-ribbon tableau $\mathcal{Q}(w)$.

Column reading Read columns from top to bottom, left to right: $1\,32\,43\,54\,6$.

Row reading Read rows from left to right, top to bottom: $56\,44\,33\,12$.

Both give words w such that $\mathcal{Q}(w)$ is the original QRT.

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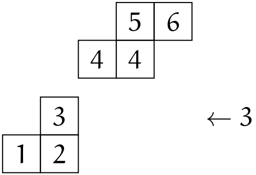

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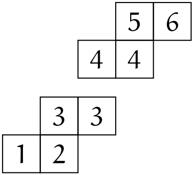

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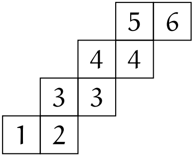

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Hypoplactic monoid

Define $\equiv$ on A^* by $u \equiv v \iff \mathcal{Q}(u) = \mathcal{Q}(v)$.

Theorem (Novelli)

The relation $\equiv$ is a congruence on A^*.

The factor monoid $H_n = A^*/{\equiv}$ is the hypoplactic monoid of rank n

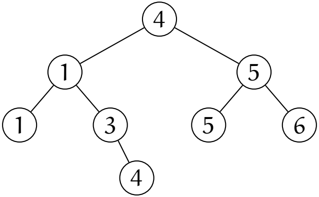
- H_n is a quotient of P_n.

Theorem (C, Gray, Malheiro 2015)

H_n is FCRS and BIAUTO.

Binary search trees

Binary search tree (BST):



To insert x into a BST T:

- Add x as a leaf node in the unique position that yields a BST.

For a word $w = w_k w_{k-1} \cdots w_1$.

- Start with an empty BST and insert w_1, then w_2, . . . , finally w_n.
- Call the resulting BST $\mathcal{T}(w)$.

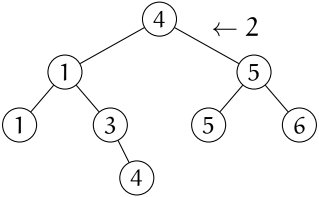
Reading of T Any word such that $\mathcal{T}(w) = T$. Equivalently, any word made up of symbols in T, with children before parents.

Readings of the example include

124315654	421565314
651425314	654231154

Binary search trees

Binary search tree (BST):



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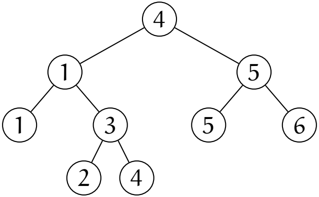

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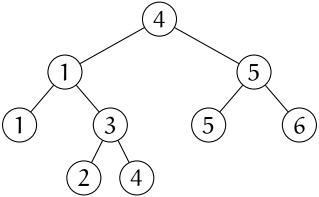

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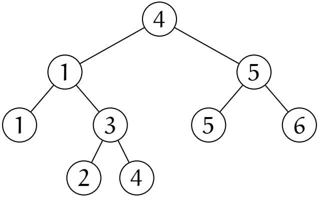

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Sylvester monoid

Define $\equiv$ on A^* by $u \equiv v \iff \mathcal{T}(u) = \mathcal{T}(v)$.

Theorem (Hivert et al. 2005)

The relation $\equiv$ is a congruence on A^*.

The factor monoid $S_n = A^*/{\equiv}$ is the sylvester monoid of rank n

Theorem (C, Gray, Malheiro 2015)

S_n admits a regular infinite complete rewriting system and is BIAUTO.

Homogeneous and content-preserving presentations

A monoid presentation $\langle A \mid \mathcal{R} \rangle$ is

Homogeneous if $|u| = |v|$ for all $(u, v) \in \mathcal{R}$;

Multihomogeneous if $|u|_a = |v|_a$ for all $(u, v) \in \mathcal{R}$ and $a \in A$.

- Plactic, hypoplactic, and sylvester monoids are multihomogeneous:

$$P_n = \langle A \mid \mathcal{P} \rangle;$$

$$H_n = \langle A \mid \mathcal{P} \cup \mathcal{H} \rangle;$$

where

$$\mathcal{P} = \{(acb, cab) : a \leqslant b < c\} \cup \{(bac, bca) : a < b \leqslant c\}$$

$$\mathcal{H} = \{(cadb, acbd), (bdac, dbca) : a \leqslant b < c \leqslant d\}$$

$$S_n = \langle A \mid (caub, acud), a \leqslant b < c \leqslant d, u \in A^* \rangle.$$

- Chinese monoids are multihomogeneous, and are BIAUTO and FCRS.
- Homogeneous monoids have solvable word problem, because all words representing an element have the same length.

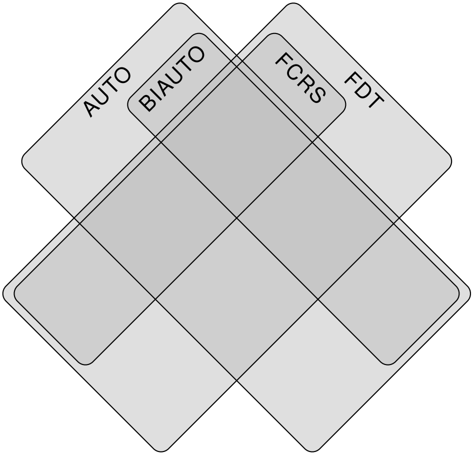
Homogeneous and content-preserving presentations

What is the relationship between FCRS, FDT, AUTO, BIAUTO in the class of homogeneous monoids?

For general monoids:

- FCRS $\implies$ FDT
- BIAUTO $\implies$ AUTO
- The properties are otherwise independent.

FCRS, FDT, AUTO, BIAUTO for homogeneous monoids



$\mathcal{M}_1$: AUTO, non-BIAUTO, FCRS, FDT (C, Gray, Malheiro).

$\mathcal{M}_2$: Reverse of $\mathcal{M}_1$. Non-AUTO, non-BIAUTO, FCRS, FDT (C, Gray, Malheiro).

$\mathcal{M}_3$: Constructed by Katsura & Kobayashi, who showed it is FDT and non-FCRS. Also BIAUTO and thus AUTO (C, Gray, Malheiro).

$\mathcal{M}_4$: BIAUTO, AUTO, non-FCRS, non-FDT (C, Gray, Malheiro).

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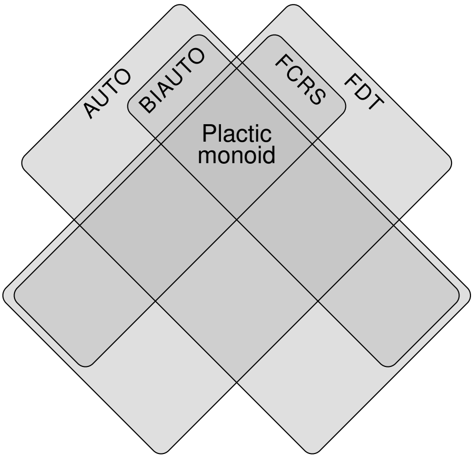

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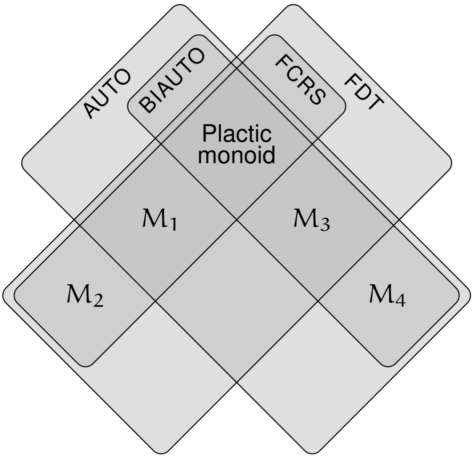

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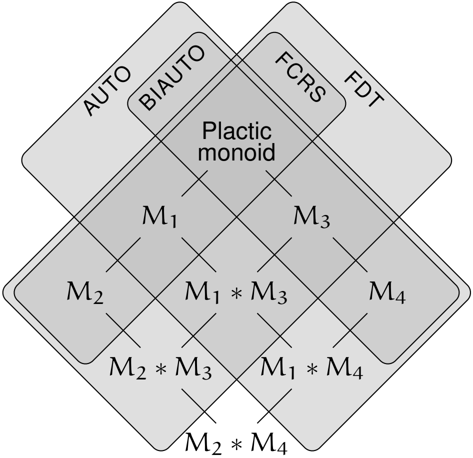

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Two concepts of conjugacy

o-conjugacy is the relation

$$x \sim_o y \iff (\exists g, h \in M)(xg = gy \wedge hx = yh).$$

primary conjugacy is the relation

$$x \sim_p y \iff (\exists u, v \in M)(x = uv \wedge y = vu).$$

- For groups, these are the usual conjugacy relation.
- For monoids, $\sim_p$ is not in general transitive.
- $\sim_p^* \subseteq \sim_o$ $\qquad$ $[\sim_p^* = \bigcup_{i=0}^{\infty} \sim_p^i]$

Theorem (Narendran & Otto)

$\sim_o$ is undecidable for FCRS monoids.

Theorem (C, Malheiro)

$\sim_o$ is undecidable for homogeneous FCRS monoids.

Conjugacy in plactic-like monoids

For $w \in A^*$, define

$$\text{ev}(w) = \big(|w|_1, |w|_2, \ldots, |w|_n\big)$$

and

$$u \sim_e v \iff \text{ev}(u) = \text{ev}(v).$$

In a multihomogeneous monoid M

$$u \sim_o v \implies (\exists g \in M)(gu = vg) \implies \text{ev}(u) = \text{ev}(v),$$

so $\sim_o \subseteq \sim_e$.

We have $\sim_p^* = \sim_o = \sim_e$ in:

- P_n [Lascoux & Schützenberger 1981]
- H_n [easy consequence of P_n result]
- S_n [C, Malheiro]
- Chinese monoid of rank n [Cassaigne et al. 2001]

Conjugacy in P_n, H_n, S_n

Theorem (Choffrut & Mercaş 2013)
$\sim_p^{\leqslant 2n-2} = \sim_o = \sim_e$ in P_n.

Theorem (C, Malheiro)
$\sim_p^{\leqslant n-1} = \sim_o = \sim_e$ in H_n. Furthermore, $\sim_p^{\leqslant n-2} \subsetneq \sim_o$.

Theorem (C, Malheiro)
$\sim_p^{\leqslant n-1} = \sim_o = \sim_e$ in S_n. Furthermore, $\sim_p^{\leqslant n-2} \subsetneq \sim_o$.

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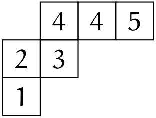
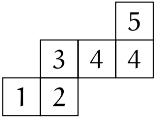
Conjugacy in the hypoplactic monoid H_5





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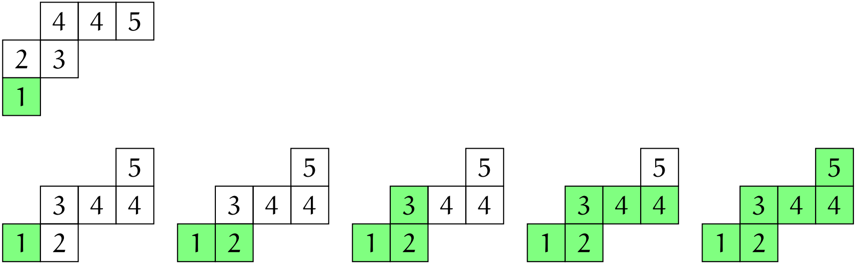
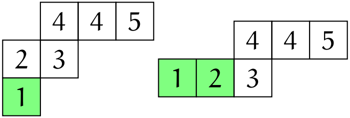
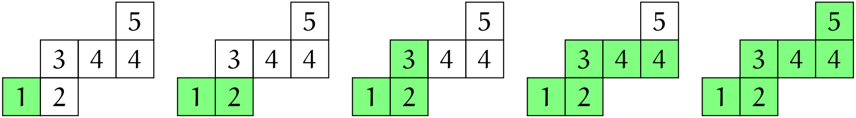


Conjugacy in the hypoplactic monoid H_5

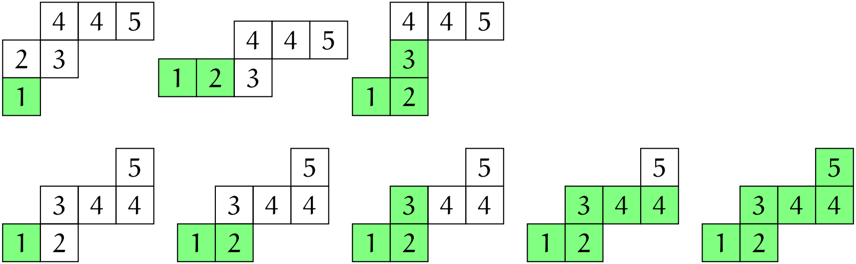
$445231 \quad \sim_p 144523$





Conjugacy in the hypoplactic monoid H_5

$$
\begin{aligned}
445231 \quad &\sim_p 144523 \\
&=_{H_5} 124345 \quad \sim_p 434512
\end{aligned}
$$



Conjugacy in the hypoplactic monoid H_5

$$
\begin{aligned}
445231 \quad &\sim_p 144523 \\
&=_{\mathrm{H}_5} 124345 \quad \sim_p 434512 \\
&\qquad\qquad\qquad =_{\mathrm{H}_5} 445312 \quad \sim_p 312445
\end{aligned}
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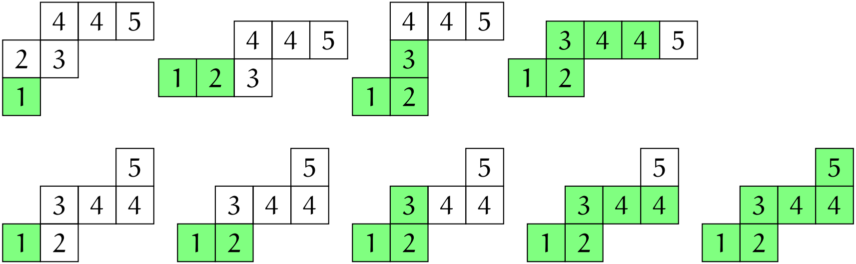
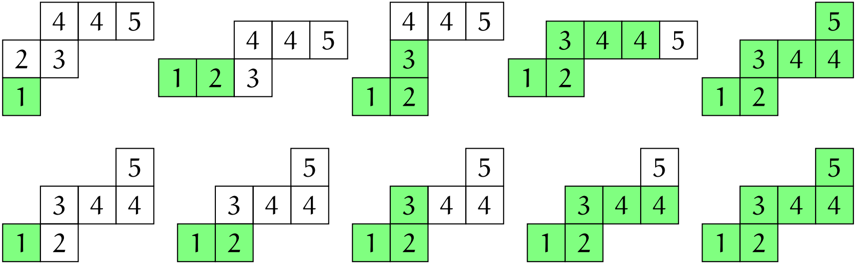
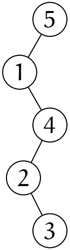


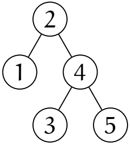
Conjugacy in the hypoplactic monoid H_5

$$
\begin{aligned}
445231 &\sim_p 144523 \\
&=_{\mathrm{H}_5} 124345 \sim_p 434512 \\
&=_{\mathrm{H}_5} 445312 \sim_p 312445 \\
&=_{\mathrm{H}_5} 132445 \sim_p 513244
\end{aligned}
$$



Conjugacy in the sylvester monoid S_5





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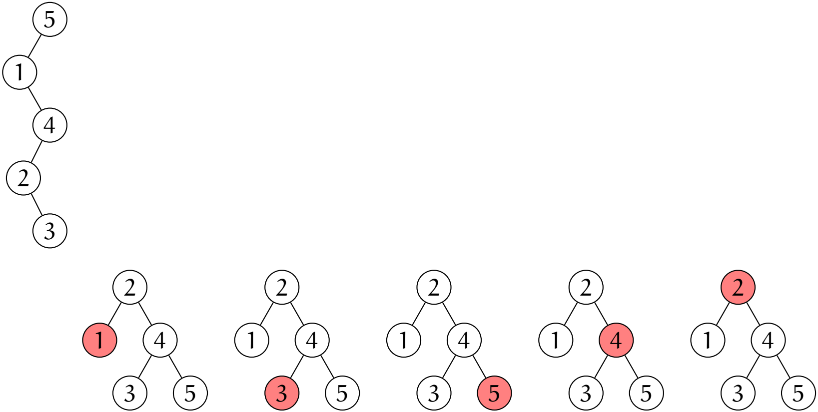

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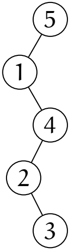
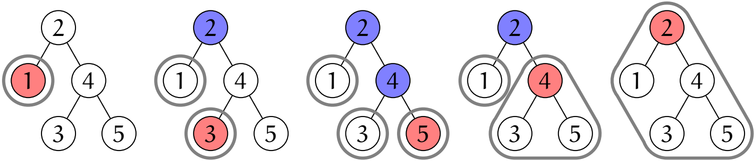
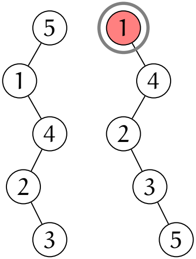
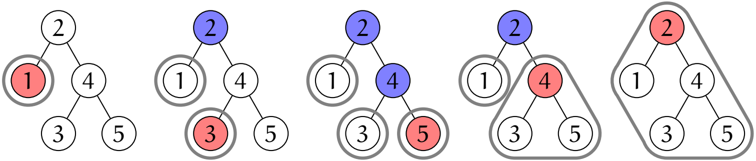




Conjugacy in the sylvester monoid S_5

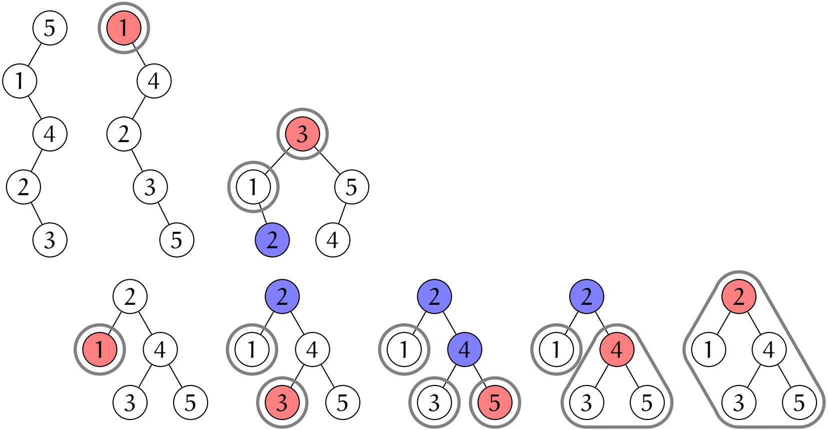
$32415 \sim_p 53241$





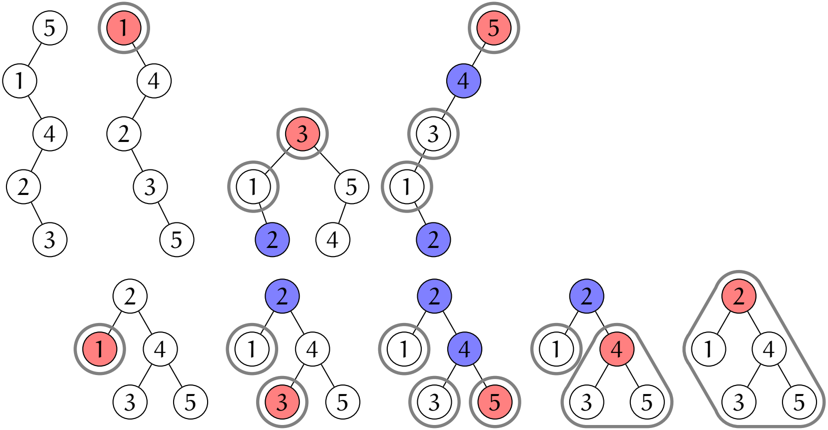
Conjugacy in the sylvester monoid S_5

$$32415 \sim_p 53241$$
$$=_{S_5} 53241 \qquad \sim_p 24153$$



Conjugacy in the sylvester monoid S_5

$$
\begin{aligned}
32415 &\sim_p 53241 \\
&=_{S_5} 53241 \sim_p 24153 \\
&\qquad\qquad\quad =_{S_5} 45213 \sim_p 21345
\end{aligned}
$$



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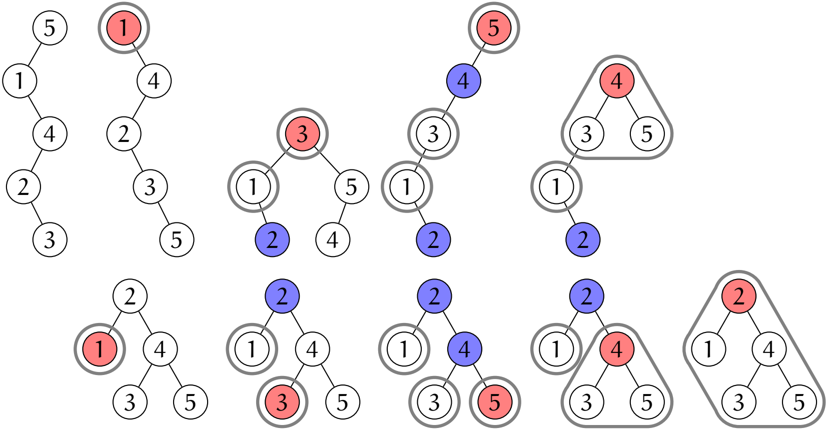
&\qquad =_{S_5} 45213 \sim_p 21345 \\
&\qquad\qquad =_{S_5} 21345 \sim_p 52134
\end{aligned}
$$



Conjugacy in the sylvester monoid S_5

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\begin{aligned}
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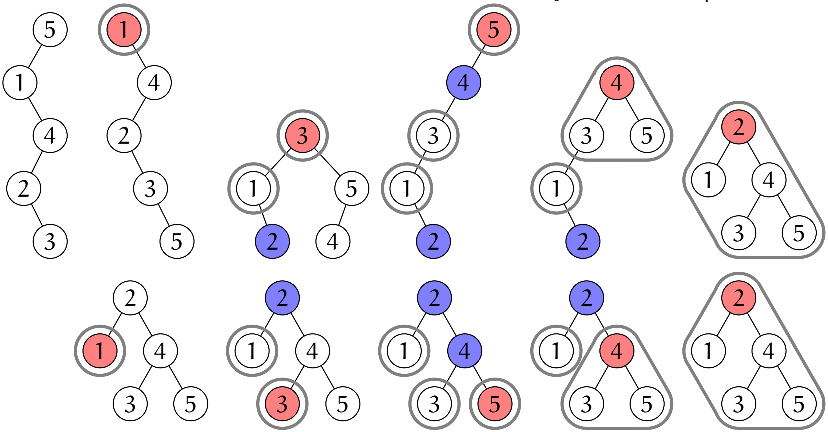
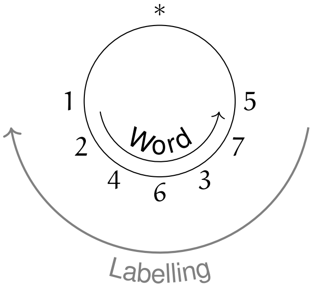
&=_{S_5} 45213 \sim_p 21345 \\
&=_{S_5} 21345 \sim_p 52134 \\
&=_{S_5} 21354 \sim_p 13542
\end{aligned}
$$



Lower bounds and cocharge sequences

To calculate the cocharge sequence of 1246375:



- Label 1 with 0.
- Having labelled i with k, proceed clockwise to $i+1$.
  - If $*$ is passed, label $i+1$ with k.
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- The cocharge sequence comprises the labels of $1, 2, \ldots$.

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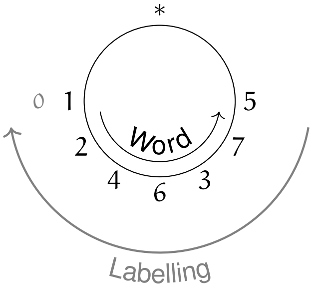

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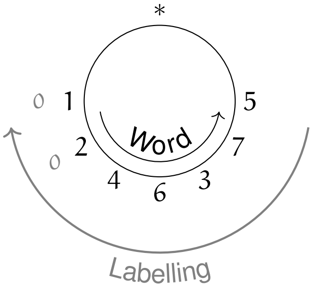

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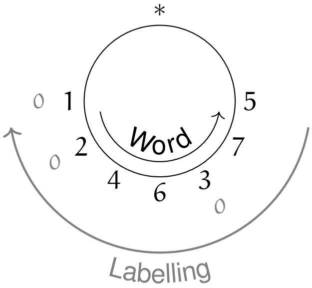

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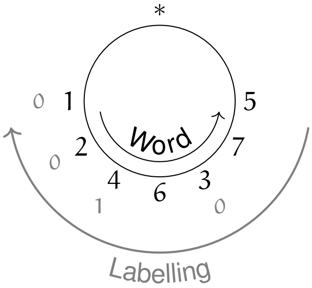

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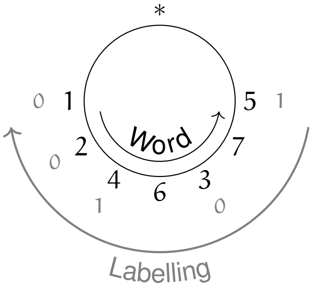

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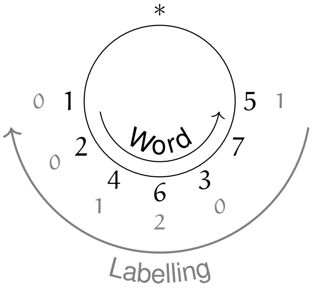

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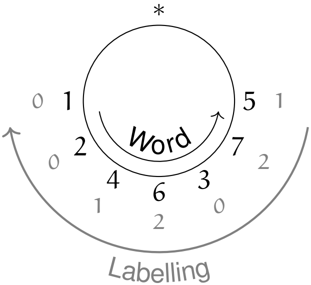

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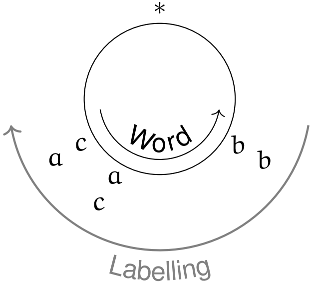
- The cocharge sequence comprises the labels of $1, 2, \ldots$.

So cochseq$(1246375) = (0, 0, 0, 1, 1, 2, 2)$.

Lower bounds and cocharge sequences

$$S_n = \langle A \mid (cavb, acvb) : a \leqslant b < c, v \in A^* \rangle.$$



Lemma

If $u =_{S_n} v$, then
$\mathrm{cochseq}(u) = \mathrm{cochseq}(v)$

Lower bounds and cocharge sequences

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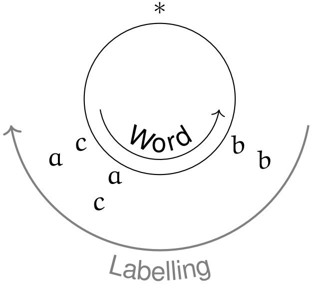


Lemma

If $u =_{S_n} v$, then $\text{cochseq}(u) = \text{cochseq}(v)$

Lemma

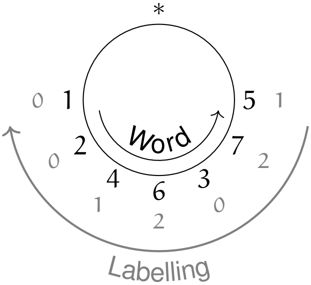
If $u =_{H_n} v$, then $\text{cochseq}(u) = \text{cochseq}(v)$

Lemma

If $u =_{P_n} v$, then $\text{cochseq}(u) = \text{cochseq}(v)$

Lower bounds and cocharge sequences

What is the effect on a cocharge sequence of applying $\sim_p$ to a word?



cochseq(1246375)
$= (0, 0, 0, 1, 1, 2, 2)$

Lower bounds and cocharge sequences

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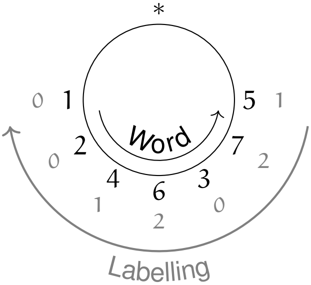
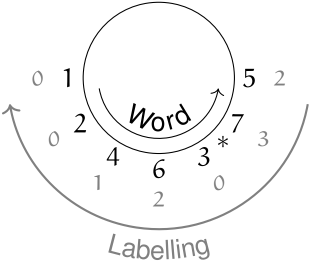




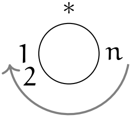
cochseq(1246375)
$= (0, 0, 0, 1, 1, 2, 2)$

cochseq(7512463)
$= (0, 0, 0, 1, 2, 2, 3)$

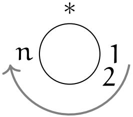
Lemma

Applying $\sim_p$ increases or decreases each term of a cocharge sequence by at most 1.

Lower bounds and cocharge sequences

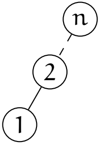
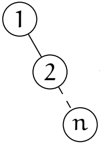


cochseq$(12\cdots n) = (0, 0, \ldots, 0)$

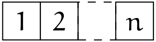
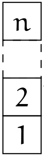


cochseq$(n\cdots 21) = (0, 1, \ldots, n-1)$

In S_n, at least $n-1$ applications of $\sim_p$ separate

$\mathcal{T}(12\cdots n) =$  and $\mathcal{T}(n\cdots 21) =$  .

In H_n, at least $n-1$ applications of $\sim_p$ separate

$\mathcal{Q}(12\cdots n) =$  and $\mathcal{Q}(n\cdots 21) =$  .

Plactic monoid P_n

Question

What is the minimum k_n such that $\sim_p^{\leqslant k_n} = \sim_o = \sim_e$ in P_n?

- Current best bounds: $n-1 \leqslant k_n \leqslant 2n-3$.
- Computer searches suggest $k_n = n-1$.
- Checked for $n \leqslant 9$ for words with no repeated symbols.

References

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'On finite complete rewriting systems, finite derivation type, and automaticity for homogeneous monoids'.
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