Complexity of Reachability, Mortality and Freeness Problems for Matrix Semigroups

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Outline of the talk

- Introduction
  - Complexity classes P, NP, PSPACE & hardness
  - Computability and undecidability
- Algorithmic problems for matrix semigroups
  - Reachability (membership)
    - Mortality
    - Identity
  - Freeness
- Open Problems
- Connections between semigroup theory, combinatorics on words and matrix problems

Computability & Complexity

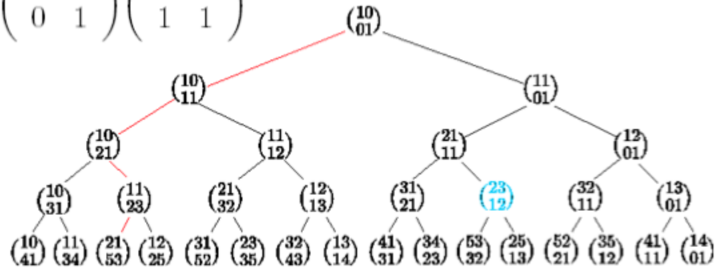
- Decidable
  - P
  - NP (NP-hard, NP-complete, ...)
  - PSPACE
- Undecidable

- Decidability: giving an algorithm which always halts and gives the correct answer in a finite time.
  - Complexity - showing equivalence of existing NP-hard, PSPACE-hard problems or analysing properties of the problem.
- Undecidability: simulation (reduction) of a Turing or Minsky machine, Post's Correspondence Problem (PCP), Hilbert's tenth problem, other undecidable problem, etc.

Marix Semigroups (Example 1)

- Given a set of finite matrices $G = \{M_1, M_2, \ldots, M_k\} \subseteq \mathbb{K}^{n \times n}$, we are interested in algorithmic decision questions regarding the semigroup S generated by G, denoted $S = \langle G \rangle$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



Decision Problems for Matrix Semigroups

- Given a matrix semigroup S generated by a finite set $G = \{M_1, M_2, \ldots, M_k\} \subseteq \mathbb{K}^{n \times n}$ (where $\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}\}$):
  - Decide whether the semigroup S
    - contains the zero matrix (Mortality Problem)
    - contains the identity matrix (Identity Problem)
    - is free (Freeness Problem)
    - is bounded, finite, etc.
  - Vector reachability problems:
    - Given two vectors x and y. Decide whether the semigroup S contains a matrix M such that $Mx = y$
    - Variants of such problems are important for probabilistic and quantum automata models

Early Reachability Results

- The MORTALITY PROBLEM was one of the earliest undecidability results of reachability for matrix semigroups

Theorem ([Paterson 70])

The MORTALITY PROBLEM *is undecidable over* $\mathbb{Z}^{3\times 3}$

- *holds even when the semigroup is generated by just* 6 *matrices over* $\mathbb{Z}^{3\times 3}$, *or for* 2 *matrices over* $\mathbb{Z}^{15\times 15}$ *[Cassaigne et al., 14]*

- The undecidability results use a reduction of Post's Correspondence Problem (PCP).

Post's Correspondence Problem

- Posts Correspondence Problem (PCP) is a useful tool for proving undecidability.

Theorem

- *PCP(2) is decidable [Ehrenfeucht, Karhumäki, Rozenberg, 82]*
- *PCP(7) is undecidable [Matiyasevich, Sénizergues, 96]*
- *PCP(5) is undecidable [Neary 15].*

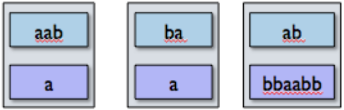


Figure : An instance of PCP(3)

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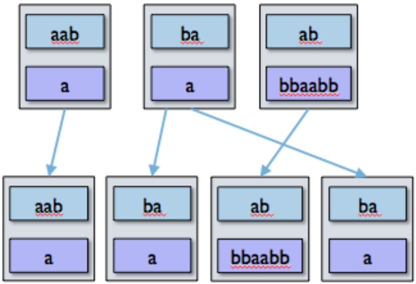


Figure : A solution - aabbaabba

Word Encodings

- Words over a binary alphabet can be encoded into 2×2 matrices
- Given a binary alphabet $\Sigma = \{a, b\}$, let $\gamma : \Sigma^* \mapsto \mathbb{Z}^{2\times 2}$ be defined by:

$$\gamma(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

  then γ is a monomorphism (injective homomorphism)
- This gives us a way to embed problems on words into problems for semigroups (for example with the direct sum)

Word Encodings (2)

- Let $\sigma(a) = 1, \sigma(b) = 2$ and $\sigma(uv) = 3^{|v|}\sigma(u) + \sigma(v)$ for every $u, v \in \Sigma^*$. Then σ is a monomorphism $\Sigma^* \to \mathbb{N}$.
- We may then define a mapping $\tau : \Sigma^* \times \Sigma^* \mapsto \mathbb{Z}^{3\times 3}$

$$\tau(u, v) = \begin{pmatrix} 1 & \sigma(v) & \sigma(u) - \sigma(v) \\ 0 & 3^{|v|} & 3^{|u|} - 3^{|v|} \\ 0 & 0 & 3^{|u|} \end{pmatrix}$$

- We can prove that $\tau(u_1, v_1) \cdot \tau(u_2, v_2) = \tau(u_1u_2, v_1v_2)$ for all $u_1, u_2, v_1, v_2 \in \Sigma^*$, thus τ is a monomorphism.
- Note that $\tau(u, v)_{1,3} = 0$ if and only if $u = v$.
- With some more work this technique can be used to show the undecidability of the Mortality Problem via a reduction of PCP, see [Cassaigne et al. 14] for example.

An aside - Skolem's Problem

- Determining if a matrix in a finitely generated matrix semigroup contains a zero in the top right element is referred to as the ZRUC (zero-in-the-right-upper-corner problem).

Definition (Linear Recurrence Sequence)

Given a sequence of recurrence coefficients $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$ and a sequence of initial values $u_0, u_1, \ldots, u_{n-1} \in \mathbb{Z}$, a linear recurrence sequence (of depth n) may be written in the form (for $k \geq n$):

$$u_k = a_{n-1}u_{k-1} + a_{n-2}u_{k-2} + \ldots + a_0 u_{k-n}.$$

An aside - Skolem's Problem

- **(Very difficult) Open Problem 1:** - For a linear recurrence sequence $u = (u_k)_{k=0}^{\infty} \subseteq \mathbb{Z}$, the zero set of u is given by $Z(u) = \{i \in \mathbb{N} | u_i = 0\}$. Determine if $Z(u)$ is an empty set.
- It is known that $Z(u)$ is a semilinear set [Skolem, 34], [Mahler, 35], [Lech, 53], and that the problem is decidable when the depth is 4 or below [Vereshchagin, 85].
- It is not difficult to show that this problem is equivalent to the following: given a matrix $M \in \mathbb{Z}^{(n+2)\times(n+2)}$, determine if there exists $k > 0$, such that $M^k_{1,(n+2)} = 0$
  - i.e. the ZRUC problem for a semigroup generated by a single matrix.

Mortality over Bounded Languages

Theorem (B., Halava, Harju, Karhumäki, Potapov, 2008)

Given integral matrices $X_1, X_2, \ldots, X_k \in \mathbb{Z}^{n \times n}$, it is algorithmically undecidable to determine whether there exists a solution to the equation:

$$X_1^{i_1} X_2^{i_2} \cdots X_k^{i_k} = Z,$$

where Z denotes the zero matrix and $i_1, i_2, \ldots, i_k \in \mathbb{N}$ are unknowns.

To prove this theorem, an encoding of Hilbert's tenth problem was used (next slide).

Mortality over Bounded Languages

Hilbert's Tenth Problem - Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

Semigroup Freeness

Definition (Code)

Let $\mathcal{S}$ be a semigroup and $\mathcal{G}$ a subset of $\mathcal{S}$. We call $\mathcal{G}$ a code if the property

$$u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$$

for $u_i, v_i \in \mathcal{G}$, implies that $m = n$ and $u_i = v_i$ for each $1 \leq i \leq n$.

Definition (Semigroup freeness)

A semigroup $\mathcal{S}$ is called free if there exists a code $\mathcal{G} \subseteq \mathcal{S}$ such that $\mathcal{S} = \mathcal{G}^+$.

- For example, consider the semigroup $\{0,1\}^+$ under concatenation. Then the set $\{00, 01, 10, 11\}$ is a code, but $\{01, 10, 0\}$ is not (since $0 \cdot 10 = 01 \cdot 0$ for example).

Matrix Freeness

Problem (Matrix semigroup freeness)

SEMIGROUP FREENESS PROBLEM - *Given a finite set of matrices $\mathcal{G} \subseteq \mathbb{Z}^{n\times n}$ generating a semigroup $\mathcal{S}$, does every element $M \in \mathcal{S}$ have a single, unique factorisation over $\mathcal{G}$? Alternatively, is $\mathcal{G}$ a code?*

- The semigroup freeness problem is *undecidable* over $\mathbb{N}^{3\times 3}$ [Klarner, Birget and Satterfield, 91]
- In fact, the undecidability result holds even over $\mathbb{N}^{3\times 3}_{uptr}$ [Cassaigne, Harju and Karhumäki, 99]

Matrix Freeness in Dimension 2

- Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 5 \\ 0 & 5 \end{pmatrix}$, is $\{A, B\}$ a code?
- Two groups of authors independently showed that in fact the following equation holds and thus the generated semigroup is not free[Gawrychowskia et al. 2010], [Cassaigne et al. 2012]:

$$AB^{10}A^2BA^2BA^{10} = B^2A^6B^2A^2BABABA^2B^2A^2BAB^2$$

  and no shorter non-trivial equation exists.
- **Open Problem 2** - Determine the decidability of the Freeness Problem over $\mathbb{N}^{2\times 2}$ (even for two matrices, or when all matrices are upper triangular).

The Identity Problem

Problem (The Identity Problem)

Given a matrix semigroup S generated by a finite set $G = \{M_1, M_2, \ldots, M_k\} \subseteq \mathbb{Z}^{n \times n}$, determine if $I_n \in \langle G \rangle$, where I_n is the n-dimensional multiplicative identity matrix.

- The IDENTITY PROBLEM is undecidable over $\mathbb{Z}^{4 \times 4}$ [B., Potapov, 2011].
- To show the undecidability of the IDENTITY PROBLEM, we introduced the Identity Correspondence Problem (next slide).

The Identity Problem - undecidability

Problem (Identity Correspondence Problem (ICP))

Identity Correspondence Problem (ICP) - Let $\Gamma = \{a, b, a^{-1}, b^{-1}\}$ *generate a free group on a binary alphabet and*

$$\Pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_m, t_m)\} \subseteq \Gamma^* \times \Gamma^*.$$

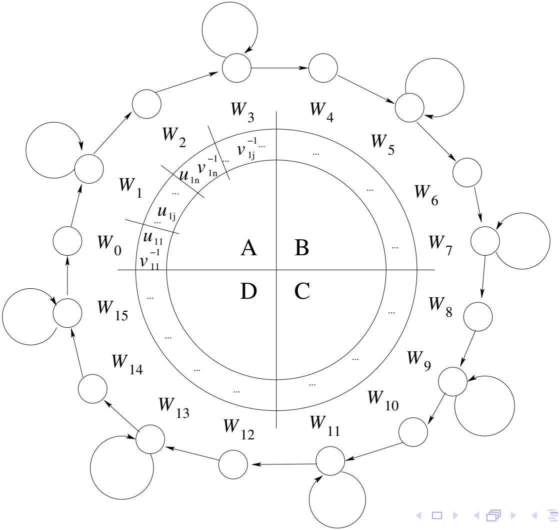
Determine if there exists a nonempty finite sequence of indices $i_1, i_2, \ldots, i_k$ *where* $1 \leq i_j \leq m$ *such that*

$$s_{i_1} s_{i_2} \cdots s_{i_k} = t_{i_1} t_{i_2} \cdots t_{i_k} = \varepsilon,$$

where ε *is the empty word (identity).*

The Identity Correspondence can be shown to be undecidable (next slides).

The Identity Problem - encoding idea



Applications of the Identity Correspondence Problem

Problem (Group Problem)

Given a free binary group alphabet $\Gamma = \{a, b, a^{-1}, b^{-1}\}$*, is the semigroup generated by a finite set of pairs of words* $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_m, v_m)\} \subset \Gamma^* \times \Gamma^*$ *a group?*

Theorem (B., Potapov, 2010)

The Group Problem *is undecidable for* $m = 8(n-1)$ *pairs of words where* n *is the minimal number of pairs for which PCP is known to be undecidable (*$n = 5$*).*

Applications of the Identity Correspondence Problem (2)

Theorem (B., Potapov, 2010)

The IDENTITY PROBLEM *is undecidable for a semigroup generated by* 48 *matrices from* $\mathbb{Z}^{4\times 4}$

- The proof uses the following injective homomorphism $\rho : \Gamma^* \to \mathbb{Z}^{2\times 2}$:

$$\rho(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \rho(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \rho(a^{-1}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \rho(b^{-1}) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

- Given an instance of the ICP - W, for each pair of words $(w_1, w_2) \in W$, define matrix $A_{w_1,w_2} = \rho(w_1) \oplus \rho(w_2)$.
- Let S be a semigroup generated by $\{A_{w_1,w_2} | (w_1, w_2) \in W\}$. Then the ICP instance W has a solution iff $I \in S$ $\square$.
- **Open Problem 3** - Determine the decidability of the IDENTITY PROBLEM over $\mathbb{Z}^{3\times 3}$.

The Identity Problem in Dimension 2

- The IDENTITY PROBLEM is decidable over $\mathbb{Z}^{2\times 2}$ [Choffrut, Karhumäki, 2005] but it is at least NP-hard [B., Potapov, 2012]
- We shall see some details of the NP-hardness proof.
- A problem is said to be NP-hard if it is at least as difficult as all other problems in the class NP (the class of problems solvable in Non-deterministic Polynomial time).

The Subset Sum Problem (SSP)

The SUBSET SUM PROBLEM is NP-hard and is a very useful tool to show other problems are also NP-hard.

Problem (Subset Sum Problem)

Given a positive integer x and a finite set of positive integer values $S = \{s_1, s_2, \ldots, s_k\}$, does there exist a (nonempty) subset of S which sums to x?

We shall now encode an instance of the subset sum problem into a set of matrices

The Structure of an Identity

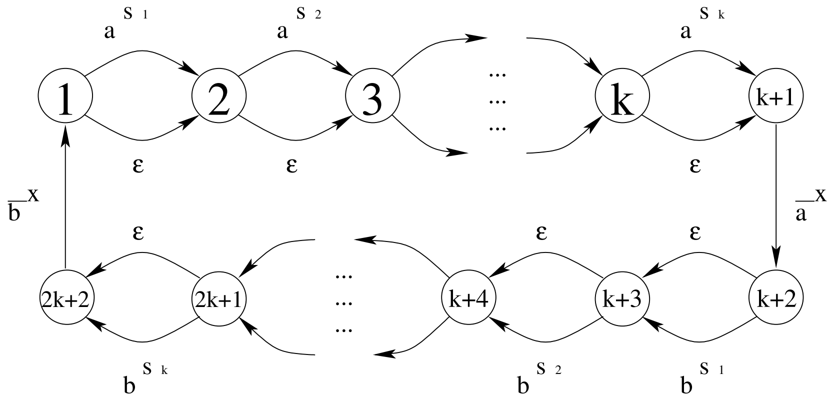


Figure : The structure of a product which forms the identity.

The Subset Sum Problem

$$W = \begin{array}{ll} \{1 \cdot a^{s_1} \cdot \overline{2}, & 1 \cdot \varepsilon \cdot \overline{2}, \\ 2 \cdot a^{s_2} \cdot \overline{3}, & 2 \cdot \varepsilon \cdot \overline{3}, \\ \vdots & \vdots \\ k \cdot a^{s_k} \cdot \overline{(k+1)}, & k \cdot \varepsilon \cdot \overline{(k+1)}, \\ (k+1) \cdot \overline{a}^x \cdot \overline{(k+2)}, & \\ (k+2) \cdot b^{s_1} \cdot \overline{(k+3)}, & (k+2) \cdot \varepsilon \cdot \overline{(k+3)}, \\ (k+3) \cdot b^{s_2} \cdot \overline{(k+4)}, & (k+3) \cdot \varepsilon \cdot \overline{(k+4)}, \\ \vdots & \vdots \\ (2k+1) \cdot b^{s_k} \cdot \overline{(2k+2)}, & (2k+1) \cdot \varepsilon \cdot \overline{(2k+2)}, \\ (2k+2) \cdot \overline{b}^x \cdot \overline{1}\} \subseteq \Sigma^*, & \end{array}$$

where $\Sigma = \{1, 2, \ldots, 2k+2, \overline{1}, \overline{2}, \ldots, \overline{(2k+2)}, a, b, \overline{a}, \overline{b}\}$ is an alphabet and $\overline{z}$ denotes z^{-1} for all alphabet characters.

The Identity Problem in Dimension 2

- We then encode the set W_2 into a set of matrices over $\mathbb{N}^{2\times 2}$ and ensure that the representation size of the matrices is polynomial in the size of the subset sum instance to complete the proof.

The Identity Problem in Dimension 2

- As a corollary, the following problems are also therefore NP-hard:
  1. Determining if the intersection of two finitely generated 2×2 integral matrix semigroups is empty.
  2. Given a finite set of 2×2 integer matrices, determining if they form a group.
  3. The ZRUC(k, 2) (zero-in-the-right-upper-corner) problem.
  4. Determining whether a finitely generated 2×2 integer matrix semigroup contains any diagonal matrix.
  5. The SCALAR/VECTOR REACHABILITY PROBLEMS over 2×2 integer matrices.

Conclusion

- We have seen a variety of problems on low dimensional, finitely generated matrix semigroups.
- Connections between combinatorics on words, automata theory and matrix semigroups.

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