One Relation Semigroups

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Statement of the Problem

Open Problem

Is the word problem soluble for every semigroup given by a single defining relation:

$$\langle a_1, \ldots, a_n \mid u = v \rangle?$$

Presentations

$$\langle a_1, \ldots, a_n \quad | \quad u_1 = v_1, \ldots, u_m = v_m \rangle$$

letters/generators | words/defining relations

The semigroup S defined: the largest/free-est semigroup generated by (copies of) $a_1, \ldots, a_k$, in which these generators satisfy all relations $u_j = v_j$ (and their consequences, but nothing else).

How to think about S: elements are words over $\{a_1, \ldots, a_n\}$; some words are equal; two words are equal iff their equality is a consequence of the defining relations.

Example

$S = \langle a, b \mid ba = a^2 b \rangle$. Every word is equal to one of the form $a^i b^j$.

Word Problem

Definition

A semigroup S with a finite generating set A has a soluble word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of S.

Example

$S = \langle a, b \mid ba = a^2 b \rangle$. One can show:

$$a^i b^j = a^k b^l \text{ in } S \Leftrightarrow i = k \ \& \ j = l.$$

Algorithm for solving the word problem: Given two words w_1, w_2 transform them into $a^i b^j$, $a^k b^l$ and then test whether $i = k$ and $j = l$.

Brief Early History and Context

- 1900 – Hilbert's 10th Problem: *Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*
- 1912 – Dehn: formulation of the word problem for groups
- 1931 – Gödel: incompleteness theorems for 1st order theories
- 1932 – Magnus: word problem for one-relator groups
- 1947 – Markov, Post: finitely presented semigroups with insoluble word problems
- 1951 – Markov: undecidability galore
- 195? – Novikov, Britton, Boone: finitely presented groups with insoluble word problems
- 1979 – Matiyasevich: negative solution to Hilbert's 10th Problem

Approaches

- Play with words (pages of induction ☹)
- Delegate (embeddings)
- Take apart (structure)
- Look at something else (other properties)

Embedding

Theorem (Magnus 1932)

Every group defined by a single relation has a soluble word problem.

Theorem (Adyan 1966)

If u and v are non-empty words which have different first letters and different last letters then the semigroup defined by $\langle a_1, \ldots, a_n \mid u = v \rangle$ embeds into the group with the same presentation, and hence has a soluble word problem.

Remark

Some descendants:

- Diagrams (Remmers 1971, 1980) and pictures (Pride 1993)
- Small overlap semigroups (Remmers)
- Applications: Kashintsev, Guba, Howie, Pride, Jackson,...

Other Types of Semigroups

Theorem (Adjan, Oganessian 1987)

One relation problem can be reduced to presentations of the type:

$$\langle a, b \mid aua = avb \rangle.$$

Corollary

If every one relation right cancellative semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.

Corollary (Ivanov, Margolis, Meakin 2001)

If every one relation inverse semigroup has a soluble word problem then every one relation semigroup has a soluble word problem.

Other Types of Semigroups

Theorem (Silva 1993)

One relation Clifford semigroups have a soluble word problem.

Question

How about completely regular semigroups?

Syntactical Approach: Special Monoids

Theorem (Adjan 1966)

Let S be the monoid defined by

$$\langle a_1, \ldots, a_n \mid u = 1 \rangle.$$

The group of units is one relator (but not necessarily same presentation). The semigroup S has a soluble word problem.

Remark

See Zhang (1992) for a short proof and generalisation.

Structure: Some Speculations

Magnus's treatment of one relator groups: Freiheitssatz, 'large' subgroup, decompose into a product of free and/or 'smaller' one-relator groups.

Theorem (Semigroup Freiheitssatz; Squier, Wrathall 1983)

Let $S = \langle a_1, \ldots, a_n \mid u = v \rangle$ be a one relation semigroup, and suppose that a_1 appears in u or v. Then the subsemigroup of S generated by $\{a_2, \ldots, a_n\}$ is free.

Problem

- *Investigate 'large' subsemigroups of one relation monoids.*
- *Candidates for large: $S \setminus \{a_1\}$; $S \setminus \langle a_1 \rangle$;...*
- *Is there a natural decomposition?*
- *Do Rees index (Ruskuc 1998) or Green index (Gray, Ruskuc, to appear) help?*

Other Properties

Investigate other structural, algebraic, combinatorial properties of one relation semigroups.

- Lallement 1974 – residual finiteness, idempotents
- Oganessian 1985 – isomorphism problem

A recent article:

A.J. Cain, V. Maltcev, Decision problems for finitely presented and one-relation semigroups and monoids, Internat. J. Algebra Comput., to appear.

What if it isn't true?

Theorem (Matiyasevich 1967)

There exists a semigroup with three defining relations which has an insoluble word problem.

Theorem (Ivanov, Margolis, Meakin 2001)

Let S be the inverse monoid defined by $\langle A \mid u = 1 \rangle$, where w is a cyclically reduced word over $A \cup A^{-1}$. Let G be the group defined by the same presentation, and let P be the submonoid of G generated by all the prefixes of u. Then S has a soluble word problem if and only if the membership problem for P is soluble.