The ordinary quiver of the algebra of the monoid of all partial functions on a set

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Monoid representations

- A representation is a (left) module over a $\mathbb{C}$-algebra A.
- A representation of a finite monoid M is a module over the monoid algebra $\mathbb{C}M$:

$$\mathbb{C}M = \{\sum_{i=1}^{|M|} \alpha_i m_i \mid \alpha_i \in \mathbb{C} \quad m_i \in M\}$$

- $\mathbb{C}M$ is a unital associative algebra.

Monoid representations

Given a finite monoid M we want to understand some invariants of the algebra $\mathbb{C}M$:

- Jacobson Radical
- Quiver
- Global dimension

Quiver of an algebra

Definition

The ordinary quiver of a finite dimensional algebra A is a directed graph defined in the following way:

- Vertices: Are in $1-1$ correspondence with the irreducible representations of A (up to isomorphism).
- Arrows: For irreducible representations N_i and N_j the number of arrows from N_i to N_j is

$$\dim e_j(\operatorname{Rad} A/\operatorname{Rad}^2 A)e_i$$

where e_i, e_j are primitive idempotents corresponding to N_i and N_j.

- Equivalently: The number of arrows equals $\dim \operatorname{Ext}^1(N_i, N_j)$.

Quiver of an algebra

Remark

The quiver of an algebra A has no arrows at all if and only if A is a semisimple algebra.

Quiver of monoid algebras

Theorem (Munn-Ponizovski)

Let M be a finite monoid with maximal group H-classes representatives $H_1, \cdots, H_n$ (one for every regular $\mathcal{J}$ class). Then there is a $1-1$ correspondence between the irreducible representations of $\mathbb{C}M$ and those of $\mathbb{C}H_1, \ldots, \mathbb{C}H_n$.

$$\operatorname{Irr} \mathbb{C}M \leftrightarrow \bigsqcup_{k=1}^{n} \operatorname{Irr} \mathbb{C}H_k$$

- *In particular, we can associate to any irreducible representation of M a specific (regular) $\mathcal{J}$ class (called its apex).*

- The apex is the lowest $\mathcal{J}$ class that the representation does not annihilate.

Quiver of monoid algebras

Hence, we can define a partial order on the irreducible representations = the vertices of the quiver.
We say that $N_i \leq N_j$ if $J_i \leq_{\mathscr{J}} J_j$ where J_i, J_j are the corresponding $\mathscr{J}$ classes.

Classical Transformation Monoids

- S_n - Symmetric group. (Permutations on n elements).
- IS_n - Inverse symmetric monoid. (Partial 1-1 maps on n elements).
- T_n - Full Transformations monoid (Functions on n elements)
- PT_n - Partial Transformations monoid (Partial functions on n elements).

Classical Transformation Monoids

Recall that if $t, s \in M$ (where $M = \mathrm{IS}_n, \mathrm{T}_n, \mathrm{PT}_n$) then:

- $t \mathscr{J} s \Leftrightarrow \mathrm{rank}(t) = \mathrm{rank}(s)$ ($|\mathrm{im}(s)| = |\mathrm{im}(t)|$).
- $t \mathscr{R} s \Leftrightarrow \mathrm{im}(t) = \mathrm{im}(s)$
- $t \mathscr{L} s \Leftrightarrow \mathrm{dom}(t) = \mathrm{dom}(s) \quad \ker(s) = \ker(t)$.

(Multiplication is from right to left)

Known results

- Putcha (1995): All the arrows in the quiver of $\mathbb{C}\,PT_n$ are going downwards.
- Putcha (1996): Computed the quiver of $\mathbb{C}\,T_n$ up to $n = 4$ (and made some observations for $n > 4$).
- Margolis, Steinberg (2012): Description of the quiver of **DO** monoids (every regular $\mathscr{D}$ class is an orthodox semigroup).

Goal

Goal of this talk

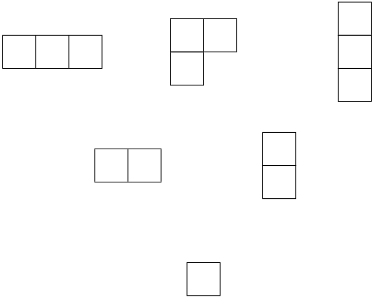
Finding the quiver of $\mathbb{C}PT_n$

Vertices of the quiver of $\mathbb{C}\,\mathrm{PT}_n$

The group $\mathscr{H}$ classes of PT_n are $S_0, \dots, S_n$ (where $S_0 \cong S_1$) so there is a correspondence between $\operatorname{Irr}\mathrm{PT}_n$ and $\displaystyle\bigsqcup_{k=0}^{n} \operatorname{Irr} S_k$.

Since the irreducible representations of S_k correspond to Young diagrams with k boxes (or partitions of k) we know the the vertices of the quiver are corresponding to the Young diagrams of with k boxes $0 \leq k \leq n$.

The **vertices** of the quiver of $\mathbb{C}\,PT_3$:



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Isomorphism between $\mathbb{C}\,\mathrm{IS}_n$ and groupoid algebra

Definition

Let G_n be the category whose objects are subsets of $\{1 \ldots n\}$, and whose morphisms are in one-to-one correspondence with elements of IS_n. For every $t \in \mathrm{IS}_n$ there is a morphism $G_n(t)$ from $\operatorname{dom} t$ to $\operatorname{im} t$, multiplication $G_n(s)G_n(t)$ is defined if and only if $\operatorname{im}(t) = \operatorname{dom}(s)$ and the result is $G_n(st)$.

G_n is a groupoid (any morphism is an isomorphism)

Isomorphism between $\mathbb{C}\,\mathrm{IS}_n$ and groupoid algebra

Theorem (Steinberg 2006)

$\mathbb{C}\,\mathrm{IS}_n \cong \mathbb{C}G_n$. *Explicit isomorphisms* $\varphi : \mathbb{C}\,\mathrm{IS}_n \to \mathbb{C}G_n$ *and* $\psi : \mathbb{C}G_n \to \mathbb{C}\,\mathrm{IS}_n$ *are defined (on basis elements) by:*

$$\varphi(s) = \sum_{t \leq s} G_n(t)$$

$$\psi(G_n(s)) = \sum_{t \leq s} \mu(t,s)t$$

Isomorphism between $\mathbb{C}\,\mathrm{IS}_n$ and groupoid algebra

$\mathbb{C}\,\mathrm{IS}_2$ is isomorphic to the algebra of the category:

$$\begin{tikzcd}
 & \{1,2\} \arrow[loop above, "{(1,2),(2,1)}"] & \\
\{1\} \arrow[loop left, "{(1,\emptyset)}"] \arrow[rr, bend left, "{(2,\emptyset)}"] & & \{2\} \arrow[loop right, "{(\emptyset,2)}"] \arrow[ll, bend left, "{(\emptyset,1)}"] \\
 & \emptyset \arrow[loop below, "{(\emptyset,\emptyset)}"] &
\end{tikzcd}$$

Definition

Let E_n be the category whose objects are the subsets of $\{1 \ldots n\}$, and whose morphisms are in one-to-one correspondence with elements of PT_n. For every $t \in \mathrm{PT}_n$ there is a morphism $E_n(t)$ from $\operatorname{dom} t$ to $\operatorname{im} t$, multiplication $E_n(s)E_n(t)$ is defined if and only if $\operatorname{im}(t) = \operatorname{dom}(s)$ and the result is $E_n(st)$.

G_n is an EI - category (any endomorphism is an isomorphism)

Isomorphism between $\mathbb{C}\,\mathrm{PT}_n$ and EI-category algebra

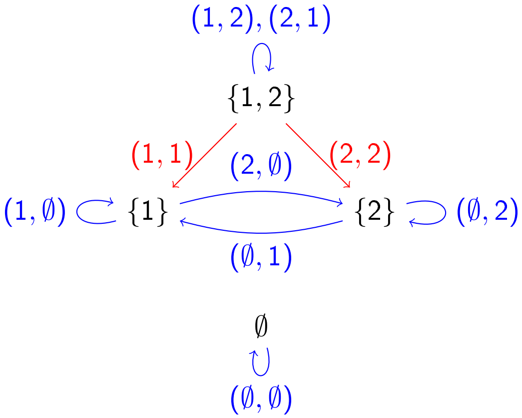
Proposition

$\mathbb{C}\,\mathrm{PT}_n \cong \mathbb{C}E_n$. *Explicit isomorphisms* $\varphi : \mathbb{C}\,\mathrm{PT}_n \to \mathbb{C}E_n$ *and* $\psi : \mathbb{C}E_n \to \mathbb{C}\,\mathrm{PT}_n$ *are defined (on basis elements) by:*

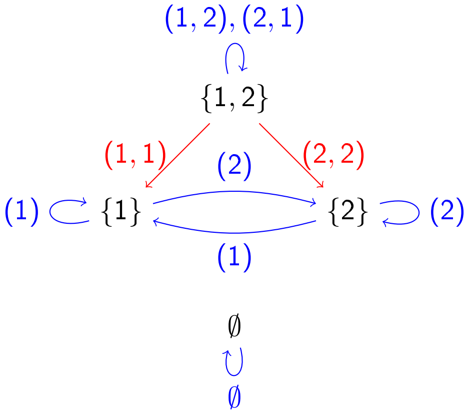
$$\varphi(s) = \sum_{t \leq s} E_n(t)$$

$$\psi(E_n(s)) = \sum_{t \leq s} \mu(t, s)t$$

$\mathbb{C}PT_2$ is isomorphic to the algebra of the category:



We can regard the morphisms of E_n as being all the **total onto** functions with dom $\subseteq \{1, \dots, n\}$.



$\mathrm{Rad}\,\mathbb{C}\,PT_n$

Remark

$\mathrm{Rad}\,\mathbb{C}\,PT_n$ is spanned by all the red arrows. In other words:

$$\mathrm{Rad}\,\mathbb{C}\,PT_n = \mathrm{span}\{E_n(t) \mid |\mathrm{dom}\ t| - |\mathrm{im}\ t| \geq 1\}$$

and more generally

$$\mathrm{Rad}^k\,\mathbb{C}\,PT_n = \mathrm{span}\{E_n(t) \mid |\mathrm{dom}\ t| - |\mathrm{im}\ t| \geq k\}$$

Skeletal EI-category algebra

Fact

If C_1 and C_2 are equivalent categories then their algebras are Morita equivalent and thus have the same ordinary quiver.

- In the category E_n, two objects (=sets) are isomorphic if and only if they are of the same size.
- It follows that the algebra of the full subcategory with objects $\{[k] \mid 0 \leq k \leq n\}$ is Morita equivalent to the algebra of E_n. ($[k] = \{1, \ldots, k\}$ for $k > 0$ and $[0] = \emptyset$)
- Denote this category by SE_n. This is a skeletal EI category.

Isomorphism between $\mathbb{C}PT_n$ and an EI-category algebra

SE_2:

$$\begin{tikzcd}
\{1,2\} \arrow[loop left, "{(1,2),(2,1)}"] \arrow[d, "{(1,1)}"'] \\
\{1\} \arrow[loop left, "(1)"] \\
\emptyset \arrow[loop left, "\emptyset"]
\end{tikzcd}$$

Isomorphism between $\mathbb{C}PT_n$ and an EI-category algebra

SE_2:

$$\begin{tikzcd}
\{1,2\} \arrow[loop left, "S_2"] \arrow[d, "{(1,1)}"'] \\
\{1\} \arrow[loop left, "S_1"] \\
\emptyset \arrow[loop left, "S_0"]
\end{tikzcd}$$

Quiver of EI-categories

Definition

A morphism f of an EI-category is called irreducible if it is not an isomorphism but whenever $f = gh$, either g is an isomorphism or h is an isomorphism.

- In SE_n the irreducible morphisms are precisely those from $[k+1]$ to $[k]$. That is:

$$\mathrm{IRR}\, SE_n([k],[r]) = \begin{cases} SE_n([k],[r]) & k = r+1 \\ \emptyset & \text{otherwise} \end{cases}$$

- Note that

$$SE_n([k],[k]) \cong S_k$$

Quiver of EI-categories

Theorem (Margolis, Steinberg (2012))

Let A be a finite skeletal EI-category and denote by Q the quiver of $\mathbb{C}A$. Then:

1. *the vertex set of Q is $\bigsqcup_{c \in A^0} \operatorname{Irr} A(c,c)$ (where $\operatorname{Irr}(G)$ is the set of irreducible modules of G).*
2. *If $V \in \operatorname{Irr}(A(c,c))$ and $U \in \operatorname{Irr} A(c',c')$, then the number of arrows from V to U is the multiplicity of $U \otimes V^*$ as an irreducible constituent in the $A(c',c') \times A(c,c)$ module $\mathbb{C} \operatorname{IRR} A(c,c')$.*

- If $V \in \text{Irr}(S_r)$ and $U \in \text{Irr}(S_k)$ are such that $r \neq k+1$ then there are no arrows from V to U since there are no irreducible morphisms between the corresponding objects in SE_n.
- The number of arrows from V to U does not depend on n.
- If $V \in \text{Irr}(S_{k+1})$ and $U \in \text{Irr}(S_k)$ then the number of arrows from V to U is the multiplicity of $U \otimes V^*$ as an irreducible constituent in the $S_k \times S_{k+1}$ module M, where M is spanned by all the onto function $f : [k+1] \rightarrow [k]$ and the operation is $(h, g) * f = hfg^{-1}$.

- By the well known theory of representations of symmetric groups, it can be shown that this multiplicity equals

$$\langle V, \mathrm{Ind}_{S_{k-1}\times S_2}^{S_{k+1}}(\mathrm{Res}_{S_{k-1}}^{S_k}(U)\otimes \mathrm{tr}_2)\rangle$$

- if α is the Young diagram corresponding to $W \in \mathrm{Irr}\, S_k$ then by the classical Branching Rule

$$\mathrm{Res}_{S_{k-1}}(W)$$

  is the sum of simple modules corresponding to the diagrams that are obtained from α by removing one box.
- If α is the Young diagram corresponding to $W \in S_{k-1}$ then

$$\mathrm{Ind}_{S_{k-1}\times S_2}^{S_{k+1}}(W \otimes \mathrm{tr}_2)$$

  is the sum of simple modules corresponding to the diagrams that are obtained from α by adding two boxes, but not in the same column.

Example

Assume $U = \ydiagram{2,1}$ and $V = \ydiagram{3,1}$. Then:

- $\mathrm{Res}^{S_3}_{S_2}(U) =$

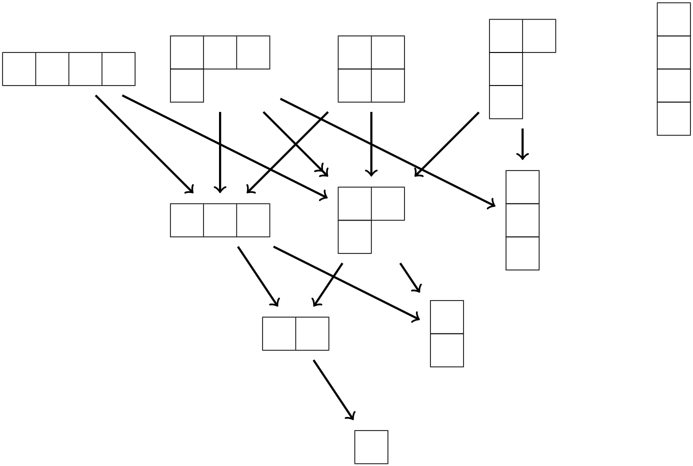
$$\ydiagram{2} + \ydiagram{1,1}$$

- $\mathrm{Ind}^{S_4}_{S_2 \times S_2}(\mathrm{Res}^{S_3}_{S_2}(U) \otimes \mathrm{tr}_2) =$

$$\ydiagram{4} + \ydiagram{3,1} + \ydiagram{2,2} + \ydiagram{3,1} + \ydiagram{2,1,1}$$

So there are two arrows from V to U.

Quiver of $\mathbb{C}PT_4$



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Quiver of $\mathbb{C}\,\mathrm{PT}_n$

Theorem (IS)

The vertices in the quiver of $\mathbb{C}\,\mathrm{PT}_n$ are in one to one correspondence with Young diagrams with k boxes where $0 \leq k \leq n$. If $\alpha \vdash k$, $\beta \vdash r$ are two Young diagrams such that $r \neq k+1$ then there are no arrows from β to α. If $r = k+1$ then there are arrows from β to α if we can construct β from α by removing one box and then adding two boxes but not in the same column. The number of arrows is the number of different ways that this construction can be carried out.

Thank you!