$\aleph_0$-categorical semigroups

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Languages

Definition

A **language** L consists of:

(i) variables, Boolean connectives $(\vee, \wedge, \neg, \rightarrow)$, quantifiers $(\exists, \forall)$, the equality symbol $=$, and parentheses (brackets and commas);

(ii) a set of function symbols, relation symbols and constant symbols .

- **Example** The language $L_S = \{\bullet\}$, where $\bullet$ is a binary function symbol.
- **Example** The language $L_R = \{R\}$, where R is a binary relation symbol.

L-sentences and L-structures

- *Well formed formulas* (wff) are inductively built from a given language. A wff in which every variable is bound (i.e. all variables are within the scope of a quantifier) is called a *sentence*.

Examples

$(\vee x(\rightarrow \exists$ is not a wff,

$\exists x\,((x \cdot x = x) \wedge (x \cdot y = y \cdot x))$ is a wff but not a sentence,

$\exists x \forall y\,((x \cdot x = x) \wedge (x \cdot y = y \cdot x))$ is a sentence.

Definition

An L-**structure** $\mathcal{M}$ is an 'interpretation' of the language L i.e. a set M equipped with functions, relations and distinguished elements (one for each function/relation/constant symbol in L).

- $\mathcal{M} = (\mathbb{Z}, \times)$ is an L_S-structure, where $\bullet^{\mathcal{M}} = \times$.
- Similarly $\mathcal{N} = (\mathbb{Z}, -)$ is also an L_S-structure, despite it not being a semigroup.

Theories and models

- If an L-sentence ϕ is true in some L-structure $\mathcal{M}$ then we say $\mathcal{M}$ has **property** ϕ.
- A set of sentences T is called a **theory**.
- An L-structure $\mathcal{M}$ is a **model** of T, denoted $\mathcal{M} \models T$, if $\mathcal{M}$ has property ϕ for all $\phi \in T$.
- The **theory of semigroups** T_S (in L_S) consists of the single sentence

  $$(\forall x)(\forall y)(\forall z)\,[x(yz) = (xy)z].$$

  Importantly, $\mathcal{M} \models T_S$ if and only if $\mathcal{M}$ is a semigroup.
- The set of all sentences true in a semigroup S is called the **theory of** S, denoted $\mathrm{Th}(S)$.

$\aleph_0$-categoricity

- A theory is $\aleph_0$-**categorical** if it has exactly one countable model, up to isomorphism. A semigroup S is $\aleph_0$-categorical if $\mathrm{Th}(S)$ is $\aleph_0$-categorical.
- This is equivalent to saying that S can be characterized, within the class of countable semigroups, by its first order properties up to isomorphism.
- Question: Which semigroups are $\aleph_0$-categorical?

$\aleph_0$-categorical Groups

- (J. Rosenstein, 1971) Any abelian group of bounded order is $\aleph_0$-categorical.
- (J. Rosenstein, 1971) The direct product of a finite number of $\aleph_0$-categorical groups is $\aleph_0$-categorical.
- (R. H. Gilman, 1984) Characterised certain characteristically simple $\aleph_0$-categorical groups.

$\aleph_0$-categoricity

- To show S is $\aleph_0$-categorical it suffices to find a list T of first order properties of S which S shares with no non-isomorphic, countable semigroup.

- **Example** The countably infinite null semigroup N, with multiplication $xy = 0$ for all $x, y \in N$, is $\aleph_0$-categorical.
  Indeed, define a theory T by

$$
\begin{aligned}
&(\forall a)(\forall b)(\forall c)[a(bc) = (ab)c], \\
&(\exists a)(\forall x)(\forall y)(xy = a), \\
&(\exists a_1)\cdots(\exists a_n)\Big(\bigwedge_{i \neq j}(a_i \neq a_j)\Big) \text{ for each } n.
\end{aligned}
$$

A rather limited approach

- However at the moment we cannot prove that a semigroup does not have such a theory.

- For example is the countably infinite monogenic semigroup $\langle a \rangle = \{a, a^2, a^3, \ldots\}$ $\aleph_0$-categorical?

- To express that a semigroup is generated by a single element we cannot write

$$(\exists x)(\forall y)(\exists n)(y = x^n)$$

or

$$(\exists x)(\forall y)(y = x^0 \vee y = x^1 \vee y = x^2 \vee \cdots).$$

- The main tool in showing $\aleph_0$-categoricity is a theorem due independently to Engeler, Ryll-Nardzewski and Svenonius, but is commonly stated as the Ryll-Nardzewski Theorem (RNT).

The Ryll-Nardzewski Theorem

- The RNT relies on the following equivalence relation on S^n;
- We say that $\underline{a} = (a_1, \ldots, a_n)$ has **the same *n*-automorphism type** as $\underline{b} = (b_1, \ldots, b_n)$ if there exists an automorphism of S sending a_i to b_i for each i.

Theorem (The Ryll-Nardzewski Theorem for semigroups)

The following are equivalent for a countable semigroup S:
i) S is $\aleph_0$-categorical;
ii) S has only finitely many n-automorphism types, for each n.

- It follows that all finite semigroups are $\aleph_0$-categorical.

A consequence of the RNT

Definition

A semigroup S is periodic if every element has finite order, that is, the monogenic subsemigroup $\langle a \rangle$ is finite for each $a \in S$.

Theorem (TQG)

If S is $\aleph_0$-categorical then S is periodic.

- Hence infinite monogenic semigroups are not $\aleph_0$-categorical, as predicted.

The Substructure of $\aleph_0$-categorical semigroups

Corollary (TQG)

Let S be an $\aleph_0$-categorical semigroup with set of idempotents E. Then the following holds

(i) $\langle E \rangle$ *is $\aleph_0$-categorical*;

(ii) *If S is an inverse semigroup, then E is an $\aleph_0$-categorical semilattice*;

(iii) *Maximal subgroups of S are $\aleph_0$-categorical*;

(iv) *S has finitely many maximal subgroups, up to isomorphism*;

Completely 0-simple semigroups

- A semigroup S is **0-simple** if $S^2 \neq \{0\}$ and if $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes.

- An idempotent e is **primitive** if

$$ef = fe = f \neq 0 \Rightarrow e = f$$

  for all $f \in E(S)$.

- A semigroup is **completely 0-simple** if it is 0-simple and contains a primitive idempotent.

Rees matrix semigroups

Theorem (The Rees Theorem)

Let G^0 *be a 0-group, let* I,Λ *be non-empty index sets and let* $P = (p_{\lambda,i})$ *be a* $\Lambda \times I$ *matrix with entries in* G^0*. Suppose no row or column of* P *consists entirely of rows (so that* P *is regular). Let* $S = (I \times G \times \Lambda) \cup \{0\}$*, and define multiplication on* S *by*

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, g\, p_{\lambda,j}\, h, \mu) & \text{if } p_{\lambda,j} \neq 0, \\ 0 & \text{if } p_{\lambda,j} = 0. \end{cases}$$

$$0(i, g, \lambda) = (i, g, \lambda)0 = 00 = 0.$$

Then S *is a completely 0-simple semigroup, known as the Rees matrix semigroup* $\mathcal{M}^0[G; I, \Lambda; P]$*. Conversely, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.*

Pure Rees matrix semigroups

Definition

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that the matrix P is over $\{0, e\}$, where e is the identity of G. Then S is called a **pure** Rees matrix semigroup. A semigroup isomorphic to a pure Rees matrix semigroups is called a **pure** completely 0-simple semigroup.

- Pure completely 0-simple semigroups generalise orthodox completely 0-simple semigroups.

Bipartite graph of Rees matrix semigroup

Definition

A *bipartite graph* is a graph whose vertices can be split into two disjoint non-empty sets L and R such that every edge connects a vertex in L to a vertex in R.

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with sandwich matrix $P = (p_{\lambda,i})_{\lambda\in\Lambda, i\in I}$. Then we may form a bipartite graph, denoted $\Gamma(P)$ with

$$\text{set of vertices } I \cup \Lambda, \tag{1}$$

$$\text{an edge } (i, \lambda) \in I \times \Lambda \text{ whenever } p_{\lambda,i} \neq 0. \tag{2}$$

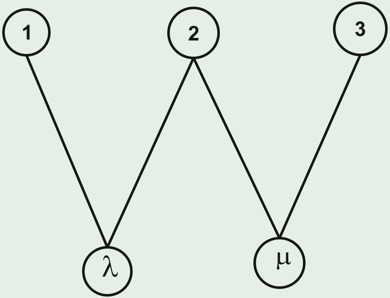
Example of a BG of Rees matrix semigroup

Example

Let $S = \mathcal{M}^0[G; \{1, 2, 3\}, \{\lambda, \mu\}; P]$ where

$$P = \begin{pmatrix} a & b & 0 \\ 0 & c & d \end{pmatrix}.$$

Then $\Gamma(P)$ is



$\aleph_0$-categorical pure Rees matrix semigroups

Theorem

Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ and $T = \mathcal{M}^0[H; J, M; Q]$ be a pair of pure Rees matrix semigroups. Then $S \cong T$ if and only if $G \cong H$ and $\Gamma(P) \cong \Gamma(Q)$.

Theorem (TQG)

The pure Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical if and only if both G and $\Gamma(P)$ are $\aleph_0$-categorical.

Theorem (C.H. Houghton)

A completely 0-simple semigroup S is pure if and only if, for all $a, b \in S$

$$[a, b \in \langle E \rangle \text{ and } a \,\mathcal{H}\, b] \Rightarrow a = b.$$

Hierarchy of results

Given a sandwich matrix P of a Rees matrix semigroup, we call an entry $p_{\lambda,i}$ of P **non-trivial** if $p_{\lambda,i} \notin \{0, e\}$.

Condition on $\mathcal{M}^0[G; I, \Lambda; P]$	Needed for $\aleph_0$-categoricity
Inverse	G
Orthodox	G and $\Gamma(P)$
Pure	G and $\Gamma(P)$
P has finitely many non-trivial entries	G and $\Gamma(P)$
I and Λ finite	G

Table: The $\aleph_0$-categoricity of certain classes of Rees matrix semigroups

Insufficiency for the general result

- **Question:** Is G and $\Gamma(P)$ being $\aleph_0$-categorical sufficient for a Rees matrix semigroup to be $\aleph_0$-categorical?
- **Counterexample:** There exists a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$ with G and $\Gamma(P)$ $\aleph_0$-categorical but such that S is not $\aleph_0$-categorical.
- **Stronger condition:** $\langle E \rangle$ $\aleph_0$-categorical implies $\Gamma(P)$ $\aleph_0$-categorical. The converse does not hold in general.
- **Open problem:** Find a non-$\aleph_0$-categorical Rees matrix semigroup such that G and $\langle E \rangle$ are both $\aleph_0$-categorical.