A representation theory approach to the rook monoid

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Outline

1. The rook monoid and its quantization
2. A basis for the rook monoid algebra
3. Techniques from the theory of cellular algebras

§ 1 The rook monoid and its quantization

The rook monoid or symmetric inverse semigroup consists of the set of all $n \times n$ matrices containing at most one entry 1 in each row and each column, and all other entries 0. The operation is matrix multiplication.

Observe that the symmetric group S_n is contained in the rook monoid ($n \times n$ permutation matrices).

For any field F, let R_n denote the rook monoid algebra. This is the F-vector space whose elements are linear combinations of elements of the rook monoid with coefficients in F. The multiplication on the monoid induces a multiplication on R_n, so R_n is a ring. Together these structures make R_n into an algebra and we can study it using representation theory.

What questions does a representation theorist ask?

Given an algebra A, we want to study A-modules (vector spaces with a compatible action of A).

- Can we classify the irreducible R_n-modules?
- Even better, could we give explicit descriptions of the irreducible R_n-modules, including giving their dimensions?
- When is R_n a semisimple algebra (every R_n-module decomposes as a direct sum of irreducibles)?
- If R_n is not semisimple, can we split it up into blocks?
- When does a block only possess finitely many indecomposable modules?
- …

Diagram notation

To avoid writing down lots of matrices, we can use diagrams to denote the elements of the rook monoid.

A diagram consists of 2 rows of n dots and some edges that join top row dots to bottom row dots. Each dot is involved in at most one edge.

Given a rook matrix x, draw an edge from dot i in the top row to dot j in the bottom if entry $x_{i,j} = 1$.

E.g. We write $x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ as the diagram

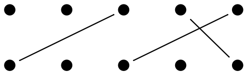
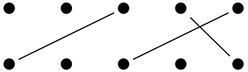
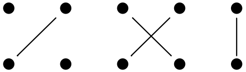
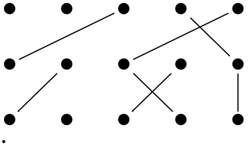

.

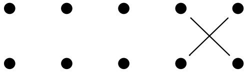
Diagram notation cont.

The product of the matrices corresponds to the concatenation of the corresponding diagrams. E.g.

$x =$  $y =$ 

then

$xy =$ 

$=$ 

The q-rook monoid algebra

More generally, we can also study the q-rook monoid algebra, defined by Solomon in 2004.

$R_n(q)$ is the associative F-algebra generated by $T_1, \ldots, T_{n-1}$ and $P_1, \ldots, P_n$ subject to the relations:

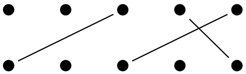
$$\begin{array}{rcll} T_i^2 & = & q.1 + (q-1)T_i, & 1 \leq i \leq n-1, \\ T_iT_{i+1}T_i & = & T_{i+1}T_iT_{i+1}, & 1 \leq i \leq n-2, \\ T_iT_j & = & T_jT_i, & |i-j| > 1, \\ T_iP_j & = & P_jT_i = qP_j, & 1 \leq i < j \leq n, \\ T_iP_j & = & P_jT_i, & 1 \leq j < i \leq n-1, \\ P_i^2 & = & P_i, & 1 \leq i \leq n, \\ P_{i+1} & = & P_iT_iP_i - (q-1)P_i, & 1 \leq i \leq n-1. \end{array}$$

The generators $T_1, \ldots, T_{n-1}$ generate the Hecke algebra $\mathcal{H}_n(q)$, a quantization of S_n.

Setting $q = 1$ we recover R_n.

§2 A basis for the rook monoid algebra

For $k = 0, 1, \ldots, n$, $A, B \subseteq \{1, 2, \ldots, n\}$ with $|A| = |B| = k$ and $w \in S_{n-k}$, let $T_{(A,B,w)}$ be the diagram with the isolated dots given by the dots of A in the top row and the dots of B in the bottom row, and the permutation w on non-isolated dots.

E.g. $x =$  has $A = \{1, 2\}$, $B = \{2, 4\}$ and $w = (2, 3)$.

More generally for $R_n(q)$, we can also define $T_{(A,B,w)}$. Say $A = \{a_1, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$, set

$$T_A = (T_{a_1-1} \cdots T_2 T_1)(T_{a_2-1} \cdots T_3 T_2) \cdots (T_{a_k-1} \cdots T_k).$$

We can define

$$T_{(A,B,w)} = i(T_A^{-1}) P_k T_w T_B^{-1}$$

(Here $w \in S_{\{k+1,\ldots,n\}}$, $T_A^{-1} T_A = 1$ and i reverses the order of the T_i generators.)

Multiplying basis elements

Define $J_k = \operatorname{span}_F\{T_{(A,B,w)} : |A| = |B| \geq k\}$. This is a 2-sided ideal of $R_n(q)$. So we have a chain of 2-sided ideals:

$$\{0\} \subseteq J_n \subseteq J_{n-1} \subseteq \cdots \subseteq J_1 \subseteq J_0 = R_n(q).$$

Within a layer it is easy to multiply: say $|A| = |B| = |C| = |D| = k$ and $w, w' \in S_{n-k}$ then in R_n,

$$T_{(A,B,w)}T_{(C,D,w')} = \begin{cases} T_{(A,D,ww')} & \text{if } B = C, \\ 0 \text{ (mod) } J_{k+1} & \text{otherwise.} \end{cases}$$

For $R_n(q)$, it is more complicated to calculate but we can still show that

$$T_{(A,B,w)}T_{(C,D,w')} = \begin{cases} q^{\left(\frac{k(k+1)}{2} - \sum_{b \in B} b\right)} T_{(A,D,ww')} & \text{if } B = C, \\ 0 \text{ (mod) } J_{k+1} & \text{otherwise.} \end{cases}$$

A little ring theory

Lemma

Let A be a ring and $J \subseteq A$ a 2-sided ideal. Then there exists a decomposition of rings $A = J \oplus J'$ with $J' \cong A/J \iff J$ has a unit element.

Proof:" $\Rightarrow$" If $A = J \oplus J'$, express $1 = e + f$ with $e \in J, f \in J'$. Then e is the required unit element of J.

" $\Leftarrow$" If J has unit element e then $e^2 = e$ and $J = eJe \subseteq eAe \subseteq J$, so $J = eAe$. Also $e(1-e) = 0$. Hence

$$A = eAe \oplus (1-e)A(1-e)$$

is the decomposition that we require. $\square$

A decomposition of the q-rook monoid algebra

Theorem

$$R_n(q) \cong \bigoplus_{k=1}^{n} J_k/J_{k+1}.$$

Proof: We simply need to find a unit element in each layer. We take

$$e_k = \sum_{A:|A|=k} \frac{1}{q^{\left(k(k+1)/2-\sum_{a\in A} a\right)}} T_{(A,A,1)} + J_{k+1}.$$

$\square$

What does this mean for $R_n(q)$-modules?

A $R_n(q)$-module is just a direct sum of modules for each layer. We need to understand the layers!

§3 Techniques from the theory of cellular algebras

In 1996 Graham and Lehrer defined a class of algebras called cellular algebras that possessed many of the nice properties of group algebras of symmetric groups or their Hecke algebras. The definition is quite technical so we omit it.

König and Xi showed how to make a new cellular algebra from an existing one via inflation (or more generally iterated inflation):
Let V be a vector space, S an algebra and $\langle\, , \,\rangle : V \times V \to S$ a bilinear form.
Define the inflation of S along V to be the vector space $V \otimes V \otimes S$ with multiplication

$$(u_1 \otimes v_1 \otimes s_1)(u_2 \otimes v_2 \otimes s_2) = u_1 \otimes v_2 \otimes s_1 \langle v_1, u_2 \rangle s_2.$$

If the bilinear form is non-degenerate then the module categories of S and the inflation $V \otimes V \otimes S$ are equivalent. (The algebras are Morita equivalent.)

The layers of $R_n(q)$ as inflations

Recall that we had a chain of ideals $\{0\} \subseteq J_n \subseteq \cdots \subseteq J_1 \subseteq J_0 = R_n(q)$, and layer k, J_k/J_{k+1} has basis elements $T_{(A,B,w)}$ with $|A| = |B| = k$.

For $k = 0, 1, \ldots, n$, let V_k be the F-vector space with basis given by all subsets $A \subseteq \{1, 2, \ldots, n\}$ of size k.

Recall the product of basis elements in layer k:

$$T_{(A,B,w)}T_{(C,D,w')} = \begin{cases} q^{\left(\frac{k(k+1)}{2} - \sum_{b \in B} b\right)} T_{(A,D,ww')} & \text{if } B = C, \\ 0 \text{ (mod) } J_{k+1} & \text{otherwise.} \end{cases}$$

So we define a bilinear form on V_k:

$$\langle B,\, C\rangle = \begin{cases} q^{\left(\frac{k(k+1)}{2} - \sum_{b \in B} b\right)} & \text{if } B = C, \\ 0 & \text{if } B \neq C. \end{cases}$$

$$\begin{aligned} J_k/J_{k+1} \quad &\cong \quad V_k \otimes V_k \otimes \mathcal{H}_{n-k}(q) \\ T_{(A,B,w)} \quad &\longleftrightarrow \quad A \otimes B \otimes T_w. \end{aligned}$$

Understanding $R_n(q)$-modules

Theorem

$R_n(q)$ is a cellular algebra, and its module category is equivalent to the module category of

$$\mathcal{H}_n(q) \oplus \mathcal{H}_{n-1}(q) \oplus \cdots \oplus \mathcal{H}_1(q) \oplus F.$$

(Or for R_n, $FS_n \oplus FS_{n-1} \oplus \cdots \oplus FS_1 \oplus F$.)

We therefore obtain the following corollaries for R_n, with similar results for $R_n(q)$:

- The irreducible R_n-modules are $V_k \otimes D^\lambda$, for $k \in 0, 1, \ldots, n$ and D^λ an irreducible FS_{n-k}-module.
- R_n is semisimple if and only if F has characteristic 0 or characteristic $p > n$.
- All questions about the representation theory of R_n are reduced to questions about the representation theory of various group algebras of symmetric groups.