VARIANTS OF SEMIGROUPS - THE CASE STUDY OF FINITE FULL TRANSFORMATION MONOIDS

IGOR DOLINKA

Abstract

This is a joint work with James East (University of Western Sydney, Australia).

Let S be a semigroup and $a \in S$. The *variant* of S with respect to a, denoted S^a, is the semigroup defined on the set S with the operation $x \star y = xay$. Variants of semigroups were introduced in 1960 by Lyapin, and various properties of variants of general semigroups were studied by Hickey in the 80s and by Khan and Lawson (2001). Particular emphasis was put (relatively recently) to variants of transformation semigroups by the Ukrainian school of semigroups, most notably by G. Tsyaputa, who classified the non-isomorphic variants and gave a characterisation of idempotents and Green's relations. Her results were described in Ganyushkin and Mazorchuk's monograph *Classical Finite Transformation Semigroups* (2009).

Here we present a comprehensive study of variants of finite full transformation semigroups including results of both algebraic and combinatorial flavour. First of all, we study variants of general semigroups and supply vital information on their regular elements and Green's relations. This approach is then applied to the case of transformation monoids. We not only recover the results of Tsyaputa, but also gain substantial information as to the complete egg-box picture of the variants, accompanied by a number of pictorial representations of the relationships of the basic structure of a full transformation monoid and its variants. It is important to stress that while the former are of course regular, this is in general not true for its variants. A single regular $\mathcal{D}$-class of a full transformation monoid $\mathcal{T}_X$ is in a variant $\mathcal{T}_X^a$ 'shattered' into a number of pieces, where exactly one of the 'shrapnels' remains regular (but with an entirely differently arranged pattern of idempotents), while all the others are non-regular and have at least one of the Green's relations trivial. The resulting $\mathcal{D}$-classes form a highly nontrivial partially ordered set, whose structure we discuss. We also compute $\operatorname{rank}(\mathcal{T}_X^a)$.

We proceed by exploring further the structure of the regular part $\operatorname{Reg}(\mathcal{T}_X^a)$, which forms a subsemigroup. We display its structure as a spined (aka pullback) product of regular semigroups of transformations with restricted range and kernel (studied recently by Sanwong and Sommanee, and Mendes-Gonçalves and Sullivan, respectively). As a result of this, $\operatorname{Reg}(\mathcal{T}_X^a)$ is obtained from $\mathcal{T}_A$ (where A is the image of a) by 'inflating' its egg-box picture so that each idempotent is replaced by a certain rectangular band, and consequently each (group) $\mathcal{H}$-class is replaced by a rectangular array (group) of $\mathcal{H}$-classes. This allows to derive a host of combinatorial results related to enumeration of equivalence classes of Green's relations, and also to compute $\operatorname{rank}(\operatorname{Reg}(\mathcal{T}_X^a))$.

Furthermore, we study and fully describe the idempotent generated subsemigroup $\mathcal{E}_X^a$ of $\mathcal{T}_X^a$, and obtain a nice generalisation of the classical result of Howie. We compute the corresponding rank and idempotent rank and show that these two are equal. We also enumerate the number of idempotent generating sets of minimal possible size. Finally, analogous results related to idempotent generation in ideals of $\operatorname{Reg}(\mathcal{T}_X^a)$ are presented.