On non-commutative frame theory

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NBSAN meeting
Edinburgh
July 21, 2014

Commutative world:

topological setting

Stone duality

A

A commutative algebra

$$\xleftrightarrow[\text{duality}]{\text{Stone-type}}$$

A^*

The dual space of A

Classical Stone duality

- A — Boolean algebra, that is A is a distributive lattice with bottom 0 and top 1 in which every element has a complement: $a \wedge a^c = 0, a \vee a^c = 1$.
- $\mathcal{P}(X)$, X a set.
- Is every Boolean algebra a powerset?
- No. There are infinitely countable Boolean algebras (e.g. finite and cofinite subsets of a countable set) but there are no infinitely countable powersets.
- Is every finite Boolean algebra a powerset?
- Yes. A — finite Boolean algebra. Then A is isomorphic to the powerset of the set of atoms of A.

Classical Stone duality

Birkhoff's representation therem, 1937

Every Boolean algebra is isomorphic to a Boolean algebra of subsets of a set.

A Boolean space is a compact Hausdorff space in which clopen sets form a base of the topology.

Theorem (M. Stone, 1937)

Every Boolean algebra is isomorphic to the Boolean algebra of clopen set of a Boolean space.

The classical Stone duality, 1937

The category of Boolean algebras is dually equivalent to the category of Boolean spaces.

Commutative world:

pointless setting

Pointless topology

For the classical topological space, the notion of a point is primary and the notion of an open set is secondary. Pointless topology studies lattices with properties similar to the properties of lattices of open sets of topological spaces.

Pointless topology studies lattices L which are

- sup-lattices: for any $x_i \in L$, $i \in I$, their join $\bigvee x_i$ exists in L.
- infinitely distributive: for any $x_i \in L$, $i \in I$, and $y \in L$

$$y \wedge (\vee_{i \in I} x_i) = \vee_{i \in I} (y \wedge x_i).$$

- Such lattices are called frames.

Frames vs locales

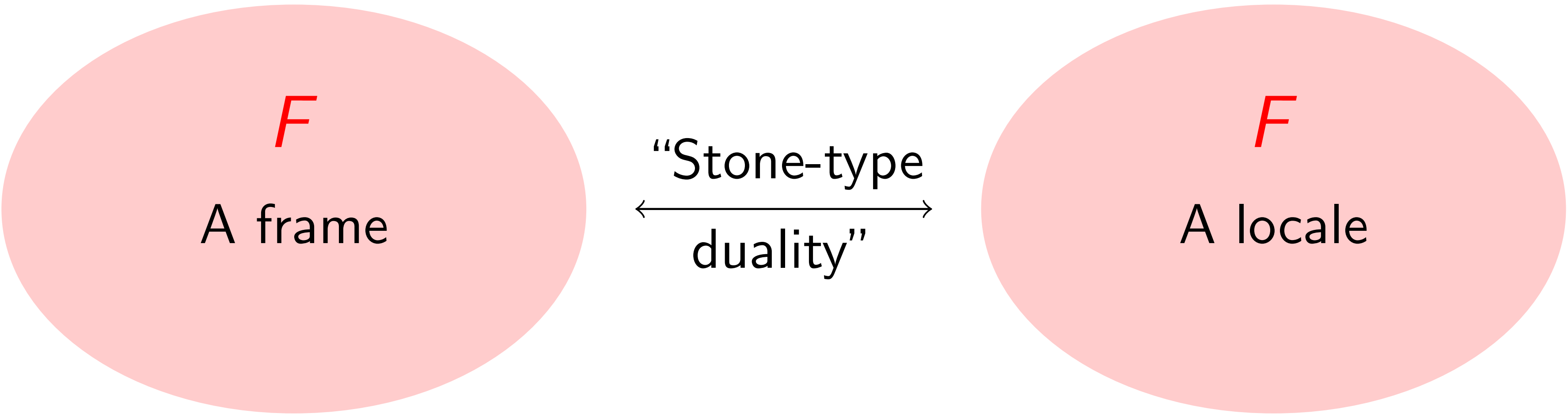
A frame morphism $\varphi : F_1 \to F_2$ is required to preserve finite meets and arbitrary joins.

The category of locales is defined to be the opposite category to the category of frames. Locales are "pointless topological spaces".

Notation

If L is a locale then the corresponding frame is sometimes denoted by $O(L)$ (frame of opens of L). By a locale morphism $\varphi : L_1 \to L_2$ we mean a frame morphism $\varphi^* : O(L_2) \to O(L_1)$ (or $\varphi^* : L_2 \to L_1$).

Frames vs locales



The adjunction

If L is a locale then poitnts of L are defined as frame morphisms $L \to \{0,1\}$. Topology of the set of points is the subspace topology from the product space $\{0,1\}^L$. This gives rise to a functor

$$\mathrm{pt} : \mathrm{Loc} \to \mathrm{Top}.$$

Conversely, assigning to a topological space its frame of opens leads to the functor

$$\Omega : \mathrm{Top} \to \mathrm{Loc}.$$

Theorem

The functor pt *is the right adjoint to the functor* Ω*.*

Spatial frames and sober spaces

A space X is sober if $\mathrm{pt}(\Omega(X)) \simeq X$. A locale F is spatial if $\Omega(\mathrm{pt}(F)) \simeq F$.

Theorem

The above adjunction restricts to an equivalence between the categories of spatial locales and sober spaces.

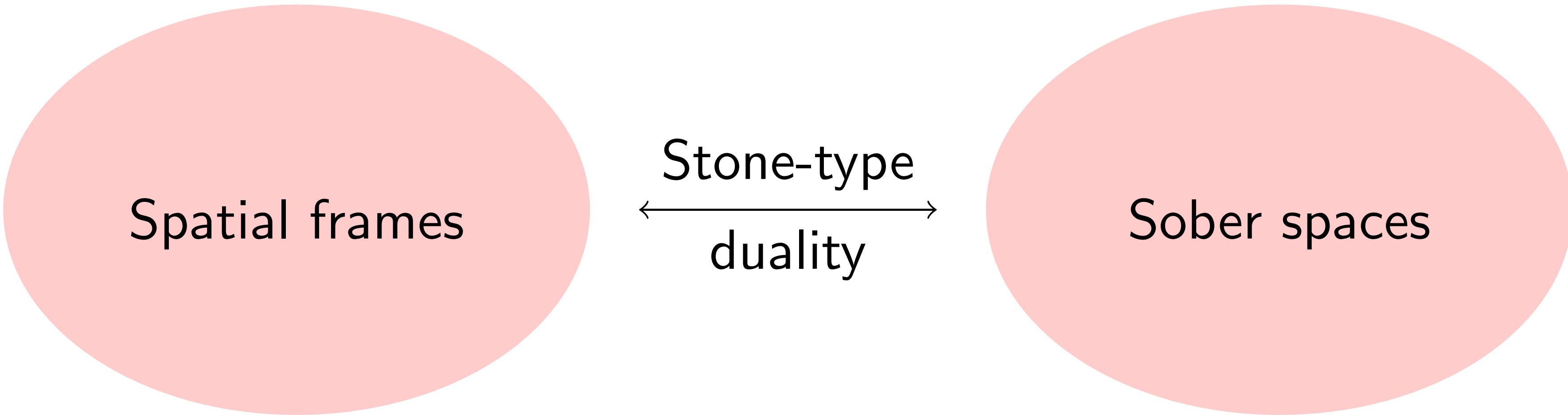
Example of a non-sober space:

$\{1, 2\}$ with indiscrete topology.

Example of non-spatial frame:

A complete non-atomic Boolean algebra, for example the Boolean algebra of Lebesgue measurable subsets of $\mathbb{R}$ modulo the ideal of sets of measure 0.

Pointset vs pointless topology



Coherent frames and distributive lattices

A space X is called spectral if it is sober and compact-open sets form a basis of the topology closed under finite intersections. A frame is called coherent if it is isomorphic to a frame of ideals of a distributive lattice.

Theorem

The following categories are pairwise equivalent:

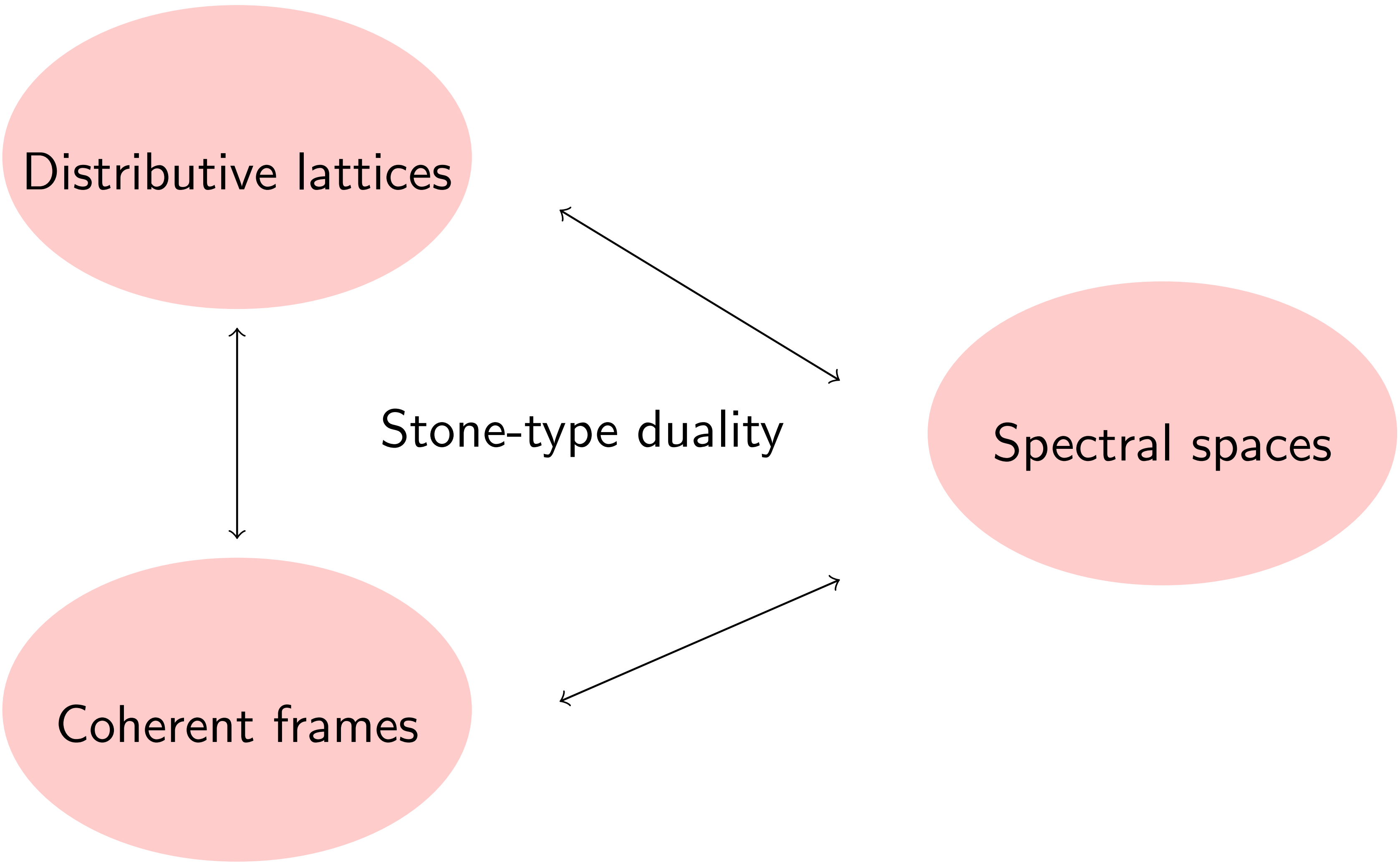
- The category of distributive lattices
- The category of coherent frames
- The category of spectral spaces

Theorem: bounded version

The following categories are pairwise equivalent:

- The category of bounded distributive lattices
- The category of coherent frames where 1 is a finite element
- The category of compact spectral spaces

Coherent frames and distributive lattices



Back to Boolean algebras

A locally compact Boolean space is a Hausdorff spectral space. A generalized Boolean algebra is a relatively complemented distributive lattice with bottom element.

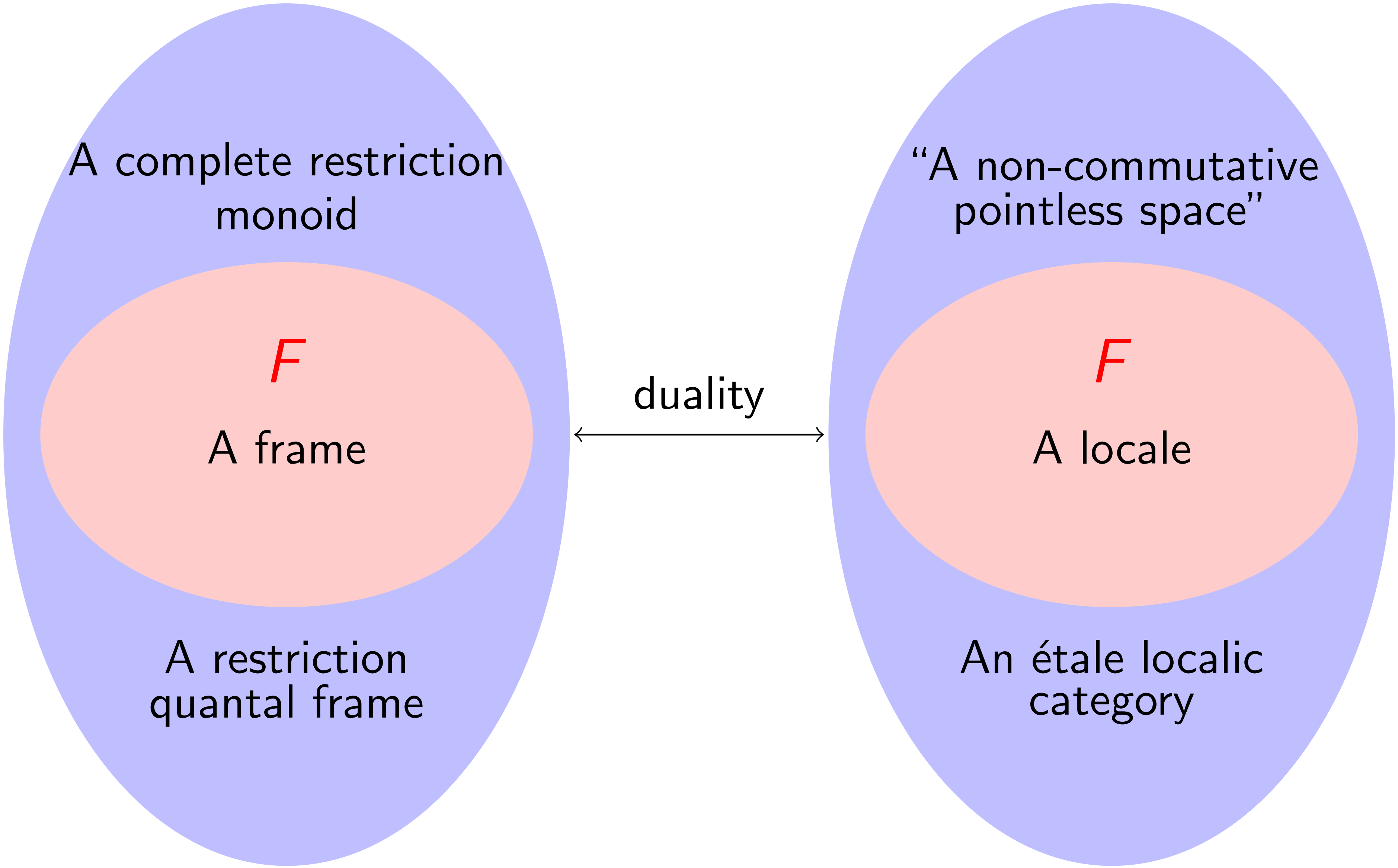
We arrive at the classical Stone duality.

Classical Stone duality

- The category of generalized Boolean algebras is dual to the category of locally compact Boolean spaces.
- The category of Boolean algebras is dual to the category of Boolean spaces.

Non-commutative world

The general idea



Quantales

A *quantale* is a sup-lattice equipped with a binary multiplication operation such that multiplication distributes over arbitrary suprema:

$$a(\vee_{i\in I} b_i) = \vee_{i\in I}(ab_i) \text{ and } (\vee_{i\in I} b_i)a = \vee_{i\in I}(b_i a).$$

A quantale is unital if there is a multiplicative unit e and involutive, if there is an involution $*$ on Q which is a sup-lattice endomorphism.

A quantal frame is a quantale which is also a frame.

Ehresmann quantal frames

Let Q be a unital quantale with unit e. We call Q an Ehresmann quantale provided that there exist two maps $\lambda, \rho : Q \to Q$ such that

(E1) both λ and ρ are sup-lattice endomorphisms;

(E2) if $a \leq e$ then $\lambda(a) = \rho(a) = a$;

(E3) $a = \rho(a)a$ and $a = a\lambda(a)$ for all $a \in Q$;

(E4) $\lambda(ab) = \lambda(\lambda(a)b)$, $\rho(ab) = \rho(a\rho(b))$ for all $a, b \in Q$.

Under multiplications, they are Ehresmann semigroups, introduced and first studied by Mark Lawson in 1991.
An Ehresmann quantal frame is an Ehresmann quantale that is a also a frame.

Partial isometries

Let Q be an Ehresmann quantal frame and $a \in S$. We say that a is a partial isometry if $b \leq a$ implies that $b = af = ga$ for some $f, g \leq e$. The set of partial isometries in Q is denoted by $\mathcal{PI}(Q)$.

Example

Let X be a non-empty set and $A \subseteq X \times X$ a transitive and reflexive relation on X. By $\mathcal{P}(A)$ we denote the powerset of A. Put $E = e^{\downarrow}$, where e is the identity relation. If $a \in \mathcal{P}(A)$, define

$$\lambda(a) = \{(x, x) \in X \times X \colon \exists y \in X \text{ such that } (y, x) \in a\}$$

and

$$\rho(a) = \{(y, y) \in X \times X \colon \exists x \in X \text{ such that } (y, x) \in a\}.$$

Then $\mathcal{P}(A)$ is an Ehresmann quantal frame. Its partial isometries are precisely partial bijections.

Localic categories

A localic category is an internal category in the category of locales.

That is, we are given

$$C = (C_1, C_0, u, d, r, m)$$

where C_1 is a locale, called the object of arrows, and C_0 is a locale, called the object of objects, together with four locale maps

$$u\colon C_0 \to C_1, \quad d, r\colon C_1 \to C_0, \quad m\colon C_1 \times_{C_0} C_1 \to C_1,$$

called unit, domain, codomain, and multiplication, respectively. $C_1 \times_{C_0} C_1$ is the object of composable pairs defined by the pullback diagram

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "r"] \\
C_1 \arrow[r, "d"'] & C_0
\end{tikzcd}$$

Localic and topological categories

The four maps u, d, r, m are subject to conditions that express the usual axioms of a category:

1. $du = ru = id$.
2. $m(u \times id) = \pi_2$ and $m(id \times u) = \pi_1$.
3. $r\pi_1 = rm$ and $d\pi_2 = dm$.
4. $m(id \times m) = m(m \times id)$.

Topological categories are defined similarly, as internal categories in the category of topological spaces. If $C = (C_1, C_0)$ is a topological category then the space of composable pairs $C_1 \times_{C_0} C_1$ equals

$$\{(a, b) \in C_1 \times C_1 \colon d(a) = r(b)\}.$$

Maps between locales

A locale map $f : L \to M$ is called semiopen if the defining frame map $f^* : M \to L$ preserves arbitrary meets. Then the left adjoint $f_!$ to f^* is called the direct image map of f.

f is called open if the Frobenius condition holds:

$$f_!(a \wedge f^*(b)) = f_!(a) \wedge b$$

for all $a \in O(L)$ and $b \in O(M)$.

Example: if $f : X \to Y$ is an open continuous map between topological spaces then it is open as a locale map.

An open locale map $f : L \to M$ is called a local homeomorphism if there is $C \subseteq O(L)$ such that

$$1_{O(L)} = \bigvee C$$

and for every $c \in C$ the frame map $O(M) \to c^{\downarrow}$ given by $x \mapsto f^*(x) \wedge c$ is surjective.

The correspondence theorem

Let Q be an Ehresmann quantal frame. It is called multiplicative if the right adjoint m^* of the multiplication map $Q \otimes_{e\downarrow} Q \to Q$ preserves arbitrary joins and thus the multiplication map is a direct image map of a locale map. It is called a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.

Let $C = (C_1, C_0, u, d, r, m)$ be an étale localic category. It is called quantal if the maps u, d, r are open and m is semiopen; and étale if u, m are open and d, r are local homeomorphisms.

Theorem (GK and Mark Lawson, 2014)

There are bijective correspondences between:

- multiplicative Ehresmann quantal frames and quantal localic categories.
- restriction quantal frames and étale localic categories.

This extends the correspondence between inverse quantal frames and étale localic groupoids due to Pedro Resende.

Restriction semigroups

Let S be a semigroup and $E \subseteq E(S)$ be a commutative subsemigroup. S is a restriction semigroup provided that there exist two maps λ, $\rho : S \to E$ such that

1. if $a \in E$ then $\lambda(a) = \rho(a) = a$;
2. $a = \rho(a)a$ and $a = a\lambda(a)$ for all $a \in S$;
3. $\lambda(ab) = \lambda(\lambda(a)b)$, $\rho(ab) = \rho(a\rho(b))$ for all $a, b \in S$;
4. $\lambda(a)b = b\lambda(ab)$, $b\rho(a) = \rho(ba)b$ for all $a, b \in S$.

Remark. Restriction semigroups form a subclass of Ehresmann semigroups. Any inverse semigroup is a restriction semigroup if one defines $\lambda(a) = a^{-1}a$ and $\rho(a) = aa^{-1}$.

$a, b \in S$ are called compatible if $a\lambda(b) = b\lambda(a)$ and $\rho(a)b = \rho(b)a$. S is called complete if E is a complete lattice and joins of compatible families of elements exist in S.

The quantalization theorem

Let S be a complete restriction monoid and $\mathcal{L}^{\vee}(S)$ be the set of all compatibly closed order ideals of S. Then it is a restriction quantal frame.

Conversely, let Q be a restriction quantal frame and $\mathcal{P}I(Q)$ be the set of partial isometries of Q. Then Q is a complete restriction monoid. This leads to the following result.

Theorem (GK and Mark Lawson, 2014)

The category of complete restriction monoids is equivalent to the category of restriction quantal frames.

This extends an equivalence between pseudogroups and inverse quantal frames established by Resende.

The duality theorem

We define morphisms of localic étale categories as functors that preserve left and right étale structure. We obtain the following result:

The duality theorem (GK and Mark Lawson, 2014)

The category of complete restriction semigroups is dual to the category of restriction quantal frames.

An example

Let X be a set and

- $X \times X$ be the pair groupoid of X.
- $\mathcal{I}(X)$ be the symmetric inverse monoid on X.
- $\mathcal{P}(X \times X)$ the powerset quantale of $X \times X$.

An observation

Either of these structures allows to recover any of the other two.

Remark

This example can be generalized if instead of $X \times X$ one starts from a reflexive and transitive relation $A \subseteq X \times X$.

The adjunction

Theorem (GK and Mark Lawson, 2014)

There is an adjunction between the category of étale localic categories and the category of étale topological categories. This adjunction extends the classical adjunction between locales and topological spaces.

Corollary

There is a dual adjunction between the category restriction quantal frames and the category of étale topological categories.

This adjunction extends the classical dual adjunction between frames and topological spaces.

The adjunction

Lawson and Lenz constructed an adjunction between pseudogroups and étale topological groupoids. It can be decomposed into the following steps:

$$S - \text{ pseudogroup};$$

$$\mathcal{L}^{\vee}(S) - \text{the enveloping inverse quantal frame of } S;$$

$$\mathcal{G}(\mathcal{L}^{\vee}(S)) - \text{the corresponding étale localic groupoid};$$

$$\mathrm{pt}(\mathcal{G}(\mathcal{L}^{\vee}(S))) - \text{the projection to an étale topological groupoid}.$$

The adjunction

In the reverse direction:

G – a topological étale groupoid;

$\Omega(G)$ – a localic étale groupoid;

$\mathcal{Q}(\Omega(G))$ – the inverse quantal frame of $\Omega(G)$;

$\mathcal{PI}(\mathcal{Q}(\Omega(G)))$ – the pseudogroup of partial isometries.

Sober and spatial categories

- Let $C = (C_1, C_0)$ be an étale localic category. Then the locale C_1 is spatial iff the locale C_0 is spatial. If these hold C is called spatial.
- Let $C = (C_1, C_0)$ be an étale topological category. Then the space C_1 is sober iff the space C_0 is sober. If these hold C is called sober.

Corollary

The category of spatial étale localic categories is equivalent to the category of sober étale topological categories.

Spectral, coherent and Boolean categories

- An étale localic category $C = (C_1, C_0)$ is called coherent (resp. strongly coherent) if the locale C_0 (resp. C_1) is coherent. This definition translates also to the corresponding restriction quantal frames.
- One can show that there is an equivalence of categories between coherent (resp. strongly coherent) étale localic categories and distributive restriction semigroups (resp. distributive restriction $\wedge$-semigroups).
- An étale topological category $C = (C_1, C_0)$ is called spectral (resp. strongly spectral) if the space C_0 (resp. C_1) is spectral.
- An étale topological category $C = (C_1, C_0)$ is called Boolean (resp. strongly Boolean) if the space C_0 (resp. C_1) is Boolean (here by Boolean space we mean a locally compact Boolean space).

The topological duality theorem

Topological duality theorem (GK and Mark Lawson, 2014)

- The category of distributive restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of spectral (resp. strongly spectral) étale topological categories.
- The category of Boolean restriction semigroups (resp. $\wedge$-semigroups) is dual to the category of Boolean (resp. strongly Boolean) étale topological categories.

Remark that the correspondence above restricts to monoids and categories with C_0 being compact.

The theorem above extends the corresponding result by Lawson and Lenz obtained for the setting of pseudogroups and topological groupoids.

Some references

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4. M. V. Lawson, D. H. Lenz, Distributive inverse semigroups and non-commutative Stone dualities, preprint, arXiv:1302.3032v1.
5. P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.