Idempotent generators in finite partition monoids

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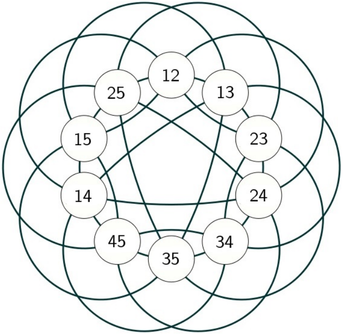
with Robert Gray (University of East Anglia)



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Joint work with Bob Gray

0. Outline
1. Transformation semigroups
   — Singular part
   — Ideals
2. Partition monoids
3. Brauer monoids
4. Jones monoids

5? Regular $*$-semigroups

Don't mention the cri%$et

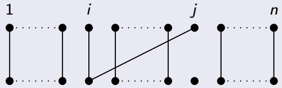
1. Transformation Semigroups

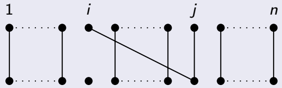
Let

- n be a positive integer
- $\mathbf{n} = \{1, \ldots, n\}$
- $\mathcal{S}_n = \{\text{permutations } \mathbf{n} \to \mathbf{n}\}$ — symmetric group
- $\mathcal{T}_n = \{\text{functions } \mathbf{n} \to \mathbf{n}\}$ — transformation semigroup
- $\mathcal{T}_n \setminus \mathcal{S}_n = \{\text{non-invertible functions } \mathbf{n} \to \mathbf{n}\}$ — singular ideal

Theorem (Howie, 1966)

- $\mathcal{T}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{T}_n \setminus \mathcal{S}_n = \langle e_{ij}, e_{ji} : 1 \leq i < j \leq n \rangle$.

$e_{ij} =$ 

$e_{ji} =$ 

Theorem (Howie, 1978)

- $\text{rank}(\mathcal{T}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{T}_n \setminus \mathcal{S}_n) = \binom{n}{2} = \frac{n(n-1)}{2}$.

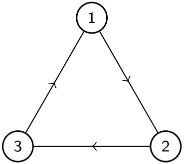
1. Transformation Semigroups

Theorem (Howie, 1978)

For $X \subseteq \{e_{ij}, e_{ji} : 1 \leq i < j \leq n\}$, define a di-graph Γ_X by

- $V(\Gamma_X) = \mathbf{n}$, and
- $E(\Gamma_X) = \big\{(i,j) : e_{ij} \in X\big\}$.

Then $\mathcal{T}_n \setminus \mathcal{S}_n = \langle X \rangle$ iff Γ_X is strongly connected and complete.



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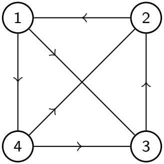

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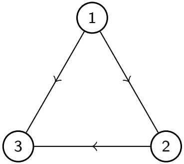


N

- $\mathcal{T}_3 \setminus \mathcal{S}_3 = \langle e_{12}, e_{23}, e_{31} \rangle$
- $\mathcal{T}_3 \setminus \mathcal{S}_3 \neq \langle e_{12}, e_{23}, e_{13} \rangle$

Theorem (Howie, 1978 and Wright, 1970)

The minimal idempotent generating sets of $\mathcal{T}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the strongly connected labelled tournaments on n nodes.

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	1	24	544	22320	1677488	236522496	$\cdots$

1. Transformation Semigroups

The ideals of $\mathcal{T}_n$ are $I_r = \{\alpha \in \mathcal{T}_n : |\operatorname{im}(\alpha)| \leq r\}$ for $1 \leq r \leq n$.

Theorem (Howie and McFadden, 1990)

If $2 \leq r \leq n-1$, then I_r is idempotent generated, and

$$\operatorname{rank}(I_r) = \operatorname{idrank}(I_r) = S(n,r),$$

a Stirling number of the second kind.

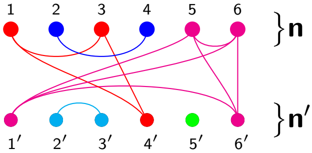
- $I_{n-1} = \mathcal{T}_n \setminus \mathcal{S}_n$ and $S(n, n-1) = \binom{n}{2}$.
- $\operatorname{rank}(I_1) = \operatorname{idrank}(I_1) = |I_1| = n$ — right zero semigroup.
- Similar results for matrix semigroups (and others).
- Today: diagram monoids.

- Let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- The *partition monoid* on $\mathbf{n}$ is

$$\mathcal{P}_n = \big\{\text{set partitions of } \mathbf{n} \cup \mathbf{n}'\big\}$$

$$\equiv \big\{\text{(equiv. classes of) graphs on vertex set } \mathbf{n} \cup \mathbf{n}'\big\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 6'\}, \{2', 3'\}, \{5'\}\Big\} \in \mathcal{P}_6$



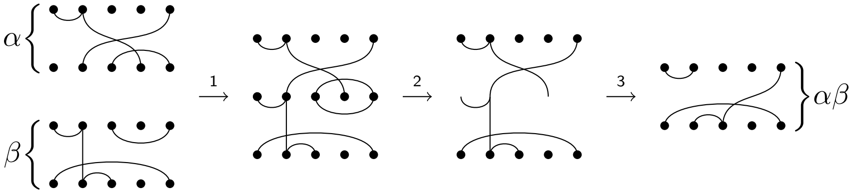
2. Partition Monoids — Product in $\mathcal{P}_n$

Let $\alpha, \beta \in \mathcal{P}_n$. To calculate $\alpha\beta$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

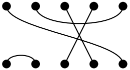
(3) smooth out resulting graph to obtain $\alpha\beta$.



The operation is associative, so $\mathcal{P}_n$ is a semigroup (monoid, etc).

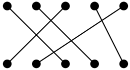
- What can we say about idempotents and ideals of $\mathcal{P}_n$?

- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{blocks of } \alpha \text{ have size } 2\}$ — Brauer monoid



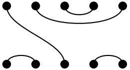
$\in \mathcal{B}_5$

- $\mathcal{S}_n = \{\alpha \in \mathcal{B}_n : \text{blocks of } \alpha \text{ hit } \mathbf{n} \text{ and } \mathbf{n}'\}$ — symmetric group



$\in \mathcal{S}_5$

- $\mathcal{J}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is planar}\}$ — Jones monoid

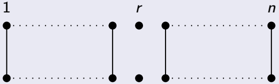


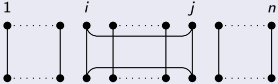
$\in \mathcal{J}_5$

What can we say about idempotents and ideals of $\mathcal{P}_n$? $\mathcal{B}_n$? $\mathcal{J}_n$?

Theorem (E, 2011)

- $\mathcal{P}_n \setminus \mathcal{S}_n$ is idempotent generated.
- $\mathcal{P}_n \setminus \mathcal{S}_n = \langle t_r, t_{ij} : 1 \leq r \leq n,\ 1 \leq i < j \leq n \rangle$.

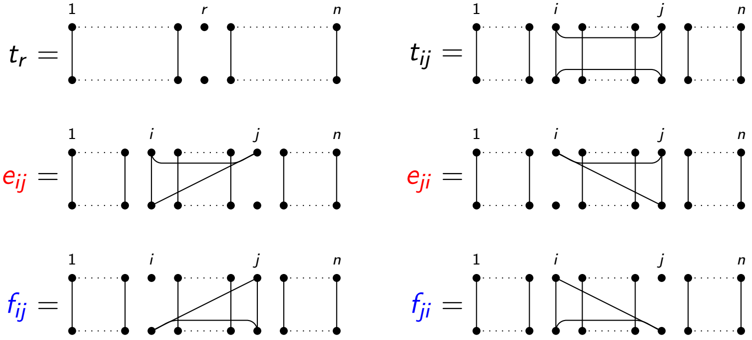
$t_r =$ 

$t_{ij} =$ 

- $\text{rank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \text{idrank}(\mathcal{P}_n \setminus \mathcal{S}_n) = \binom{n+1}{2} = \frac{n(n+1)}{2}$.

Any minimal idempotent generating set for $\mathcal{P}_n \setminus \mathcal{S}_n$ is a subset of

$$\{t_r : 1 \le r \le n\} \cup \{t_{ij}, e_{ij}, e_{ji}, f_{ij}, f_{ji} : 1 \le i < j \le n\}\,.$$



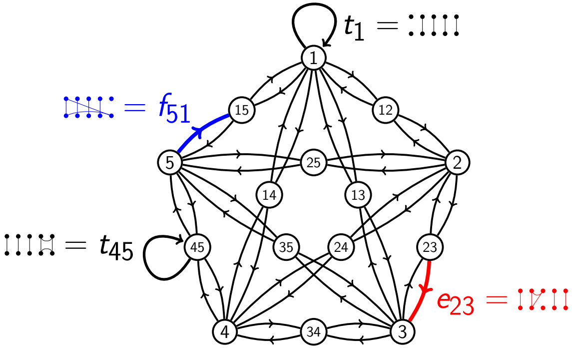
To see *which* subsets generate $\mathcal{P}_n \setminus \mathcal{S}_n$, we create a graph...

Let Γ_n be the di-graph with vertex set

$$V(\Gamma_n) = \{A \subseteq \mathbf{n} : |A| = 1 \text{ or } |A| = 2\}$$

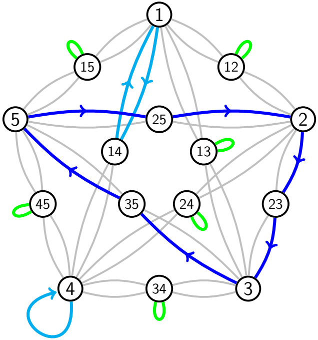
and edge set

$$E(\Gamma_n) = \{(A, B) : A \subseteq B \text{ or } B \subseteq A\}.$$



Γ_5 (with loops omitted)

A subgraph H of a di-graph G is a permutation subgraph if $V(H) = V(G)$ and the edges of H induce a permutation of $V(G)$.



A permutation subgraph of Γ_n is determined by:

- a permutation of a subset A of $\mathbf{n}$ with no fixed points or 2-cycles ($A = \{2, 3, 5\}$, $2 \mapsto 3 \mapsto 5 \mapsto 2$), and
- a function $\mathbf{n} \setminus A \to \mathbf{n}$ with no 2-cycles ($1 \mapsto 4$, $4 \mapsto 4$).

2. Partition Monoids

Theorem (E+Gray, 2013)

The minimal idempotent generating sets of $\mathcal{P}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of Γ_n.

The number of minimal idempotent generating sets of $\mathcal{P}_n \setminus \mathcal{S}_n$ is equal to

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n,n-k},$$

where $a_0 = 1$, $a_1 = a_2 = 0$, $a_{k+1} = ka_k + k(k-1)a_{k-2}$, and

$$b_{n,k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{k}{2i} (2i-1)!! n^{k-2i}.$$

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	3	20	201	2604	40915	754368	$\cdots$

2. Partition Monoids

The ideals of $\mathcal{P}_n$ are

$$I_r = \{\alpha \in \mathcal{P}_n : \alpha \text{ has} \leq r \text{ transverse blocks}\}$$

for $0 \leq r \leq n$.

Theorem (E+G, 2013)

If $0 \leq r \leq n-1$, then I_r is idempotent generated, and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \sum_{j=r}^{n} \binom{n}{j} S(j,r) B_{n-j} = \sum_{j=r}^{n} S(n,j) \binom{j}{r},$$

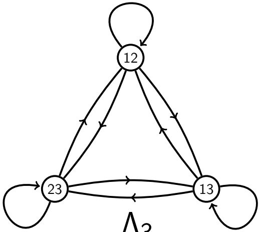
where B_k is the kth Bell number.

Let Λ_n be the di-graph with vertex set

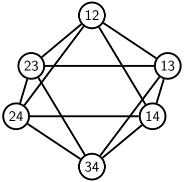
$$V(\Lambda_n) = \{A \subseteq \mathbf{n} : |A| = 2\}$$

and edge set

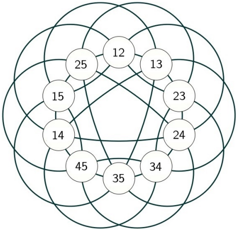
$$E(\Lambda_n) = \{(A, B) : A \cap B \neq \emptyset\}.$$



Λ_3



Λ_4



Λ_5

Theorem (E+G, 2013)

The minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ are in one-one correspondence with the permutation subgraphs of Λ_n.

No formula is known for the number of minimal idempotent generating sets of $\mathcal{B}_n \setminus \mathcal{S}_n$ (yet). Very hard!

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	1	6	265	126,140	855,966,441	????	$\cdots$

There are (way) more than $(n-1)! \cdot (n-2)! \cdots 3! \cdot 2! \cdot 1!$.

- Thanks to James Mitchell for $n = 5, 6$.
- Partition monoids are now on GAP!
- Semigroups package: `tinyurl.com/semigroups`

The ideals of $\mathcal{B}_n$ are

$$I_r = \{\alpha \in \mathcal{B}_n : \alpha \text{ has} \leq r \text{ transverse blocks}\}$$

for $0 \leq r = n - 2k \leq n$.

Theorem (E+G, 2013)

If $0 \leq r = n - 2k \leq n - 2$, then I_r is idempotent generated and

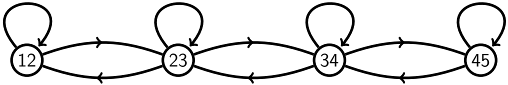
$$\text{rank}(I_r) = \text{idrank}(I_r) = \binom{n}{2k}(2k-1)!! = \frac{n!}{2^k k! r!}.$$

Let Ξ_n be the di-graph with vertex set

$$V(\Xi_n) = \big\{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\big\}$$

and edge set

$$E(\Xi_n) = \{(A,B) : A \cap B \neq \emptyset\}.$$



Ξ_5

4. Jones Monoids

Theorem (E+G, 2013)

The minimal idempotent generating sets of $\mathcal{J}_n \setminus \{1\}$ are in one-one correspondence with the permutation subgraphs of Ξ_n.

The number of minimal idempotent generating sets of $\mathcal{J}_n \setminus \{1\}$ is F_n, the nth Fibonacci number.

n	0	1	2	3	4	5	6	7	$\cdots$
	1	1	1	2	3	5	8	13	$\cdots$

4. Jones Monoids

The ideals of $\mathcal{J}_n$ are

$$I_r = \{\alpha \in \mathcal{J}_n : \alpha \text{ has} \leq r \text{ transverse blocks}\}$$

for $0 \leq r = n - 2k \leq n$.

Theorem (E+G, 2013)

If $0 \leq r = n - 2k \leq n - 2$, then I_r is idempotent generated and

$$\text{rank}(I_r) = \text{idrank}(I_r) = \frac{r+1}{n+1}\binom{n+1}{k}.$$

Values of $\text{rank}(I_r) = \text{idrank}(I_r)$:

$n \setminus r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		2		1							
4	2		3		1						
5		5		4		1					
6	5		9		5		1				
7		14		14		6		1			
8	14		28		20		7		1		
9		42		48		27		8		1	
10	42		90		75		35		9		1

5. Regular $*$-semigroups

Definition

$(S, \cdot, ^*)$ is a *regular $*$-semigroup* if $(S, \cdot)$ is a semigroup and

$$s^{**} = s, \quad (st)^* = t^* s^*, \quad ss^* s = s \quad (\text{and } s^* s s^* = s^*).$$

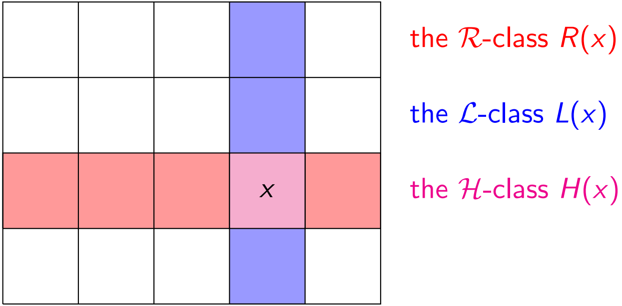
Examples

- groups and inverse semigroups, where $s^* = s^{-1}$
- $\mathcal{P}_n$, where $\alpha^* = \alpha$ turned upside down
- $\mathcal{B}_n$, $\mathcal{J}_n$, $\mathcal{S}_n$
- Not $\mathcal{T}_n$ — $\mathcal{J}$-classes must be square

Green's relations on a semigroup S are defined, for $x, y \in S$, by

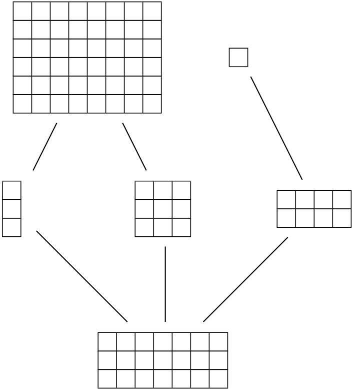
- $x\mathcal{L}y$ iff $S^1x = S^1y$,
- $x\mathcal{R}y$ iff $xS^1 = yS^1$,
- $x\mathcal{J}y$ iff $S^1xS^1 = S^1yS^1$,
- $x\mathcal{H}y$ iff $x\mathcal{L}y$ and $x\mathcal{R}y$.

Within a $\mathcal{J}$-class $J(x)$ in a finite semigroup:



The $\mathscr{J}$-classes of a semigroup S are partially ordered:

- $J(x) \leq J(y)$ iff $x \in S^1 y S^1$.



5. Regular $*$-semigroups

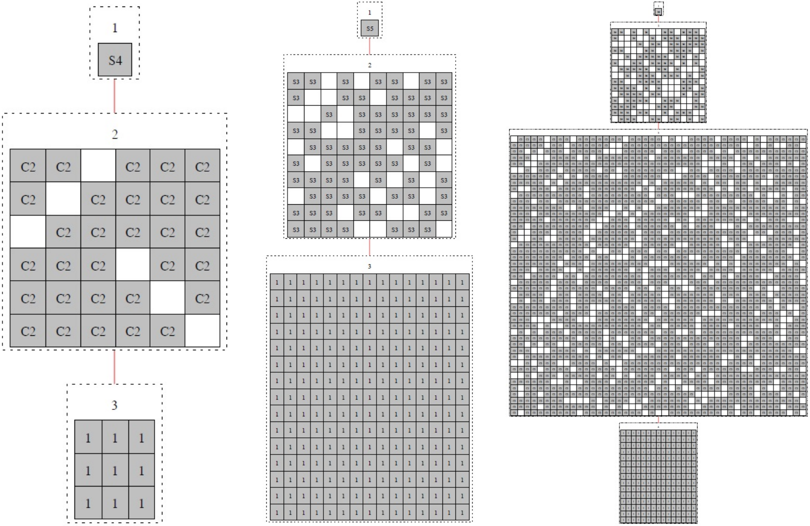
The $\mathcal{J}$-classes of a semigroup S are partially ordered:

- $J(x) \leq J(y)$ iff $x \in S^1 y S^1$.

If S is $\mathcal{P}_n \setminus \mathcal{S}_n$ or $\mathcal{B}_n \setminus \mathcal{S}_n$ or $\mathcal{J}_n \setminus \{1\}$, then:

- S is a regular $*$-semigroup,
- S is idempotent generated,
- the $\mathcal{J}$-classes form a chain $J_1 < \cdots < J_k$,
- $J_r \subseteq \langle J_{r+1} \rangle$ for each r.

5. Regular $*$-semigroups — $\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6$ (thanks to GAP)



5. Regular $*$-semigroups — ideals

Theorem (applies to $\mathcal{P}_n \setminus \mathcal{S}_n$ and $\mathcal{B}_n \setminus \mathcal{S}_n$ and $\mathcal{J}_n \setminus \{1\}$)

Let S be a finite regular $*$-semigroup and suppose

- S is idempotent generated,
- the $\mathscr{J}$-classes of S form a chain $J_1 < \cdots < J_k$,
- $J_r \subseteq \langle J_{r+1} \rangle$ for each r.

Then

- the ideals of S are the sets $I_r = \langle J_r \rangle = J_1 \cup \cdots \cup J_r$,
- the ideals of S are idempotent generated,
- $\text{rank}(I_r) = \text{idrank}(I_r) =$ the number of $\mathscr{R}$-classes in J_r.

If J is a $\mathcal{J}$-class of a semigroup S, we may form the *principle factor*

$$J^\circ = J \cup \{0\} \qquad \text{with product} \qquad s \circ t = \begin{cases} st & \text{if } s, t, st \in J \\ 0 & \text{otherwise.} \end{cases}$$

Lemma (applies to $\mathcal{P}_n \setminus \mathcal{S}_n$ and $\mathcal{B}_n \setminus \mathcal{S}_n$ and $\mathcal{J}_n \setminus \{1\}$)

If $S = \langle J \rangle$ where J is a $\mathcal{J}$-class, then

$$\text{rank}(S) = \text{rank}(J^\circ).$$

Further, S is idempotent generated iff J° is, and

$$\text{idrank}(S) = \text{idrank}(J^\circ).$$

Any minimal (idempotent) generating set for S is contained in J.

5. Regular $*$-semigroups — minimal generating sets

Proposition

Let

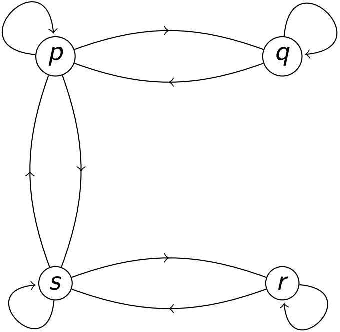
- S be a regular $*$-semigroup,
- $E(S) = \{s \in S : s^2 = s\}$ — idempotents of S,
- $P(S) = \{s \in S : s^2 = s = s^*\}$ — *projections* of S.

Then

- $E(S) = P(S)^2$,
- $\langle E(S) \rangle = \langle P(S) \rangle$,
- S is idempotent generated iff it is projection generated,
- each $\mathcal{R}$-class (and $\mathcal{L}$-class) contains exactly one projection.

Consider the projections of some finite regular $*$-semigroup J°:

p	pq		ps
qp	q		
		r	rs
sp		sr	s



$$0 = pr = rp = qr = rq = qs = sq$$

We create a graph $\Gamma(J^\circ)$.

Definition

The graph $\Gamma(J^\circ)$ has:

- vertices $P(J) = \{\text{non-zero projections}\}$,
- edges $p \to q$ iff $pq \in J$.

If $S = \langle J \rangle$ is a finite idempotent generated regular $*$-semigroup, we define $\Gamma(S) = \Gamma(J^\circ)$.

Theorem

A subset $F \subseteq E(J)$ determines a subgraph $\Gamma_F(S)$ with

$$V(\Gamma_F(S)) = P(J) \quad \text{and} \quad E(\Gamma_F(S)) = \{p \to q : pq \in F\}.$$

The set F is a minimal (idempotent) generating set for S iff $\Gamma_F(S)$ is a permutation subgraph.

Thanks for listening