From knot groups to knot semigroups

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On the picture: the logo of a shopping centre in Colchester presents what in this talk we shall treat as the generators of the semigroup of the trefoil knot

Knot groups

- 3 ways of looking at knot groups:
  - The fundamental group
  - Wirtinger presenation (1905)
  - Dehn presentation (1914?)
- Whichever way you generate a group corresponding to a knot, this is always the same infinite group

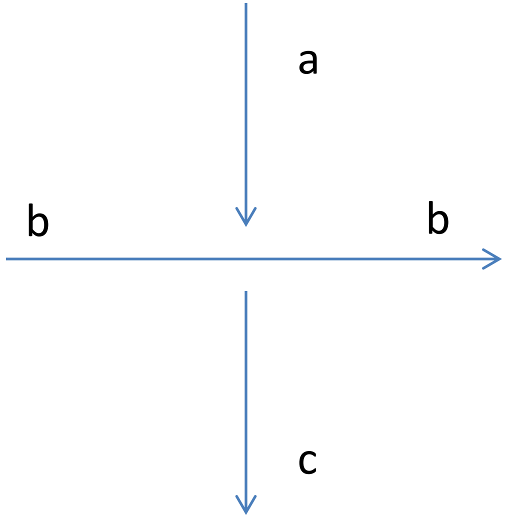
The fundamental group

- The knot is treated as a 3D labyrinth

- A fitting quotation:
  ‘Walk around me. Go ahead, walk around me. Clear around. Did you find anything?’
  ‘No. No, Steve. There are no strings tied to you, not yet.’

Wirtinger presentation

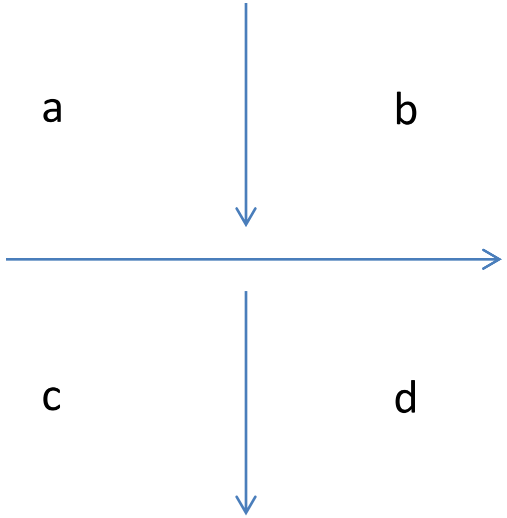
- Arcs (treated as directed arcs with a consistent orientation throughout) are considered as generators
- Each relation is 'read around' a crossing: move anti-clockwise and read out the letters on arcs coming from the right (or coming from the left, inverted)



$$abc^{-1}b^{-1} = 1$$

Dehn presentation

- Faces are considered as generators
- Each relation is an equality of two ‘ratios’ found at a crossing: treat the continuous arc as a ‘division sign’



$$ac^{-1} = bd^{-1}$$

My immediate goal

- Introduce interesting semigroups based on knots, using Wirtinger presenation and Dehn presenation as an inspiration

For comparison

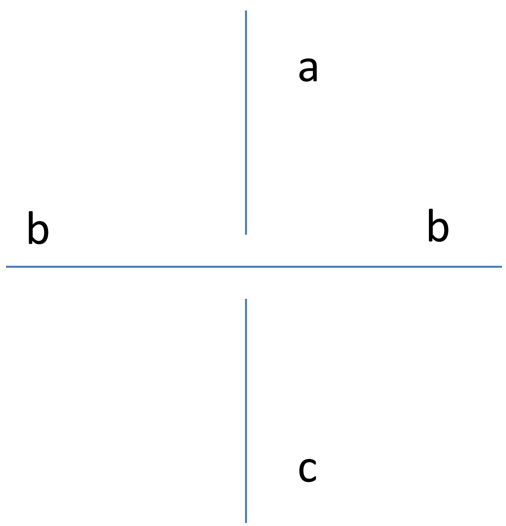
- A three-generated free commutative semigroup
- Generators: a, b, c
- Relations:
  - ab=ba
  - ac=ca
  - bc=cb

New relations

- Generators: a, b, c
- Relations:
  - ab=ca, ba=ac
  - ba=cb, ab=bc
  - ca=bc, ac=cb
- (when one letter jumps over another letter, that other letter turns into the third letter)
- Example: abc=?
- Example: aabb=bbcc

Knot semigroup

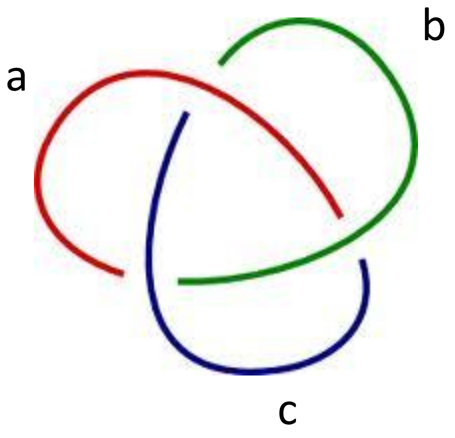
- Arcs are considered as generators
- Each relation is an equality of two products found at a crossing: read the letters in the opposite angles, clockwise in one of them and anticlockwise in the other



$$ab = bc$$

$$ba = cb$$

The semigroup of the trefoil knot



- Relations:
  - ab=ca, ba=ac
  - ba=cb, ab=bc
  - ca=bc, ac=cb

Studying the semigroup

- Only words of an equal length can be equal to each other
- There are three elements of length 1:
  a, b, c
- There are 5 elements of length 2:
  aa, bb, cc, ab, ac

Studying the semigroup

- For any length greater than 2, there are exactly 6 elements:
  a...a, b...b, c...c, a...ab, a...ac, a...abb
- For example, every word of length 6 is equal to one of the following:
  aaaaaa
  bbbbbb
  cccccc
  aaaaab
  aaaaac
  aaaabb

The 6 elements of a given length

- Question:
  How do you prove that every word is equal to a word of one of these forms?
  a...a, b...b, c...c, a...ab, a...ac, a...abb
- Answer:
  By induction on the length of the word

The 6 elements of a given length

- Question:
  How do you prove that these words are not equal to one another?
  a...a, b...b, c...c, a...ab, a...ac, a...abb
- Answer:
  It is possible to find an invariant of a word which is not changed by the relations

The invariant

- Relations:
  - ab=ca, ba=ac
  - ba=cb, ab=bc
  - ca=bc, ac=cb
- Replace letters by numbers e.g. a=0, b=1, c=2
- Consider them in arithmetic modulo 3
- Relations:
  - 01=20, 10=02
  - 10=21, 01=12
  - 20=12, 02=21

The invariant

- Relations:
  - 01=20, 10=02
  - 10=21, 01=12
  - 20=12, 02=21
- Each relation preserves the difference between the first and the second letter (modulo 3)

The invariant

- More generally, take a word, for example, abbc
- Convert to a sequence of digits modulo 3 0112
- Calculate the sum, with odd digits taken with the positive sign and even digits with the negative sign:
  +0-1+1-2=1
- Using the relations will not change this value

The semigroup of the trefoil knot

- Done:
  Now we know everything there is to know about the semigroup of the trefoil knot (based on its standard diagram)
- To do:
  the big problem of whether the semigroups changes if a knot is represented by some non-standard diagram

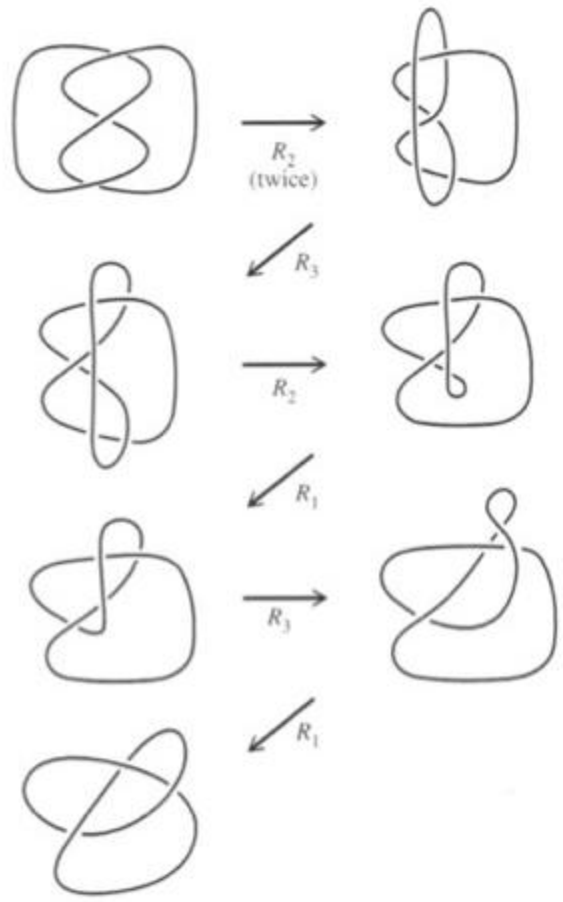


Diagram from: http://www.askamathematician.com/2011/01/q-how-does-a-scientist-turn-ideas-into-math/

Knot invariants

- We want our semigroups to be *invariants* of knots.
- That is, the semigroup should depend on the knot, but not on the specific diagram of the knot used to build the semigroup.
- For comparison, knot groups are invariants of knots

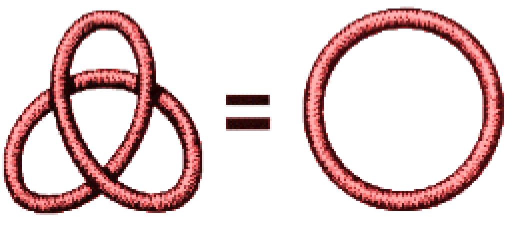


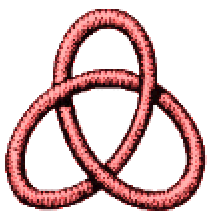
Diagram from http://www.popmath.org.uk/exhib/pagesexhib/unknum.html

Cancellation property

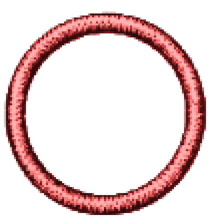
- In knot semigroups, my suggestion is
  - not to have inverses
  - but to have cancellation
- Examples with positive integers:
  - The equation $x+y=z$ cannot be solved for x
  - But the equation $x+y=z+y$ can be solved for x
- In knot semigroups,
  - if $uv=wv$ then $u=w$
  - if $vu=vw$ then $u=w$

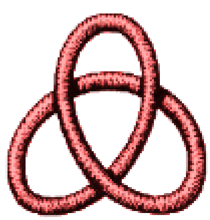
The trivial knot:
in the standard and in the trefoil-like position

For comparison: the trefoil knot
(the standard diagram):



The trivial knot, also known as unknot
(the standard diagram and a trefoil-like diagram):

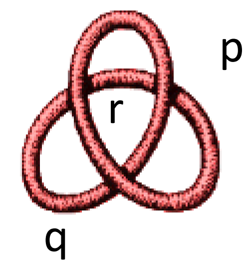




- The semigroup of the trivial knot (the standard diagram) is the infinite cyclic semigroup
- The semigroup of the trivial knot (the trefoil-like diagram) is also the infinite cyclic semigroup
- For proving the latter result, we use the cancellation property

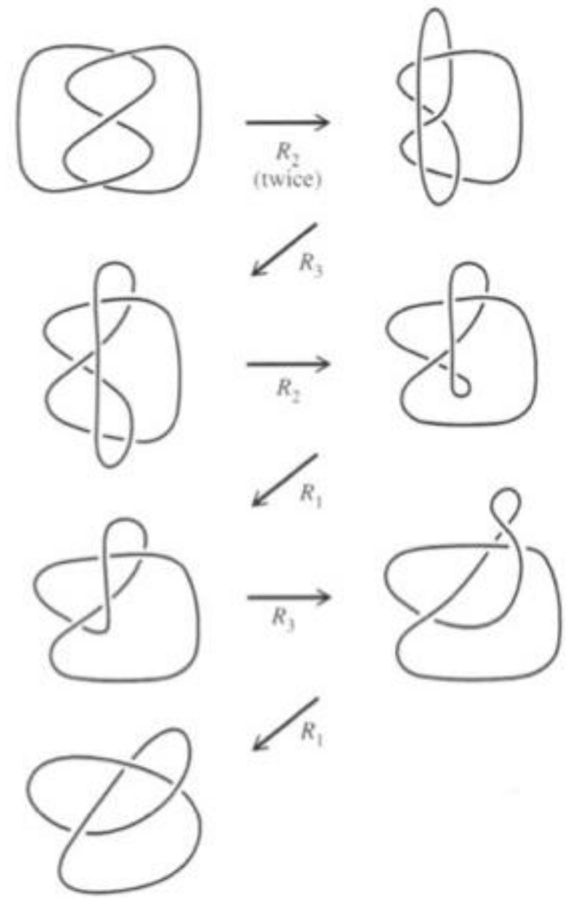
The trivial knot in the trefoil-like position

- The relations are
  - qp=pr and pq=rp
  - rp=pp and pr=pp
  - pp=pq and pp=qp
- Without cancellation, the semigroup is ‘almost’ cyclic ☹
- With cancellation, p=q=r ☺



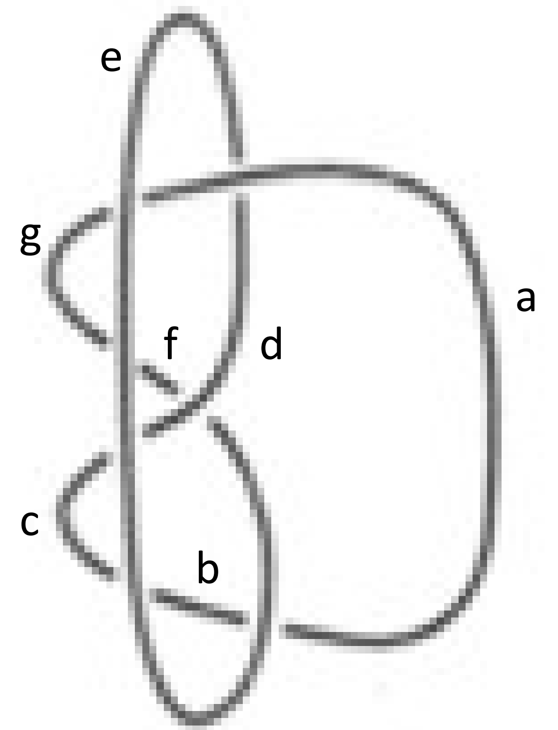
The trefoil knot in an awkward position

- All knots on this diagram are equal.
- I calculated their semigroups, and they all are isomorphic to each other.



The trefoil knot in an awkward position

- Here is one of the knot diagrams from the previous slide
- Relations are ae=eb, etc.
- The relations together with cancellation simplify the generators as follows:
  - a=c=f
  - b=d=g
  - e

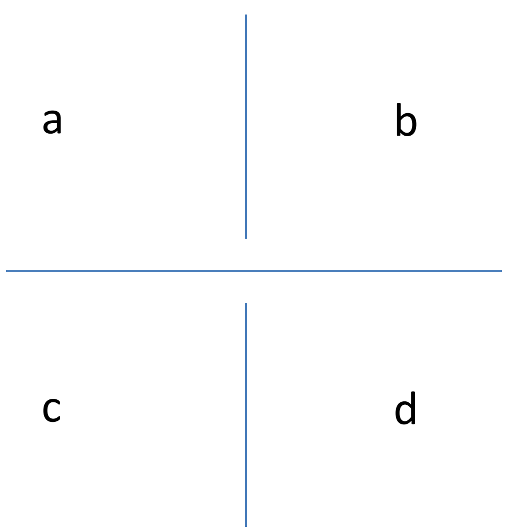


Further research

- My personal perception of knot semigroups:

Dehn-style knot semigroup

- Faces are considered as generators
- Each relation is an equality of two products ‘read around’ a crossing as you circle it clockwise or anticlockwise
- (I have not started studying this semigroup yet)



$$ab = dc$$
$$ac = db$$
$$ba = cd$$
$$bd = ca$$

Why semigroups?

- In a knot group, there are ‘too many’ sets of generators
- Isomorphism between groups is difficult to establish

- In a knot semigroup, there is only one set of generators
- Isomorphism between semigroups is easy to establish?

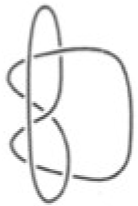
The group of the trefoil knot

- Generated by a,b with a relation $a^2 = b^3$
- Or: by x,y with a relation $xyx = yxy$
- Or (from my own non-related research): by p, q with a relation $pqp^{-1} = qp^{-1}q^{-1}$
- Establishing isomorphism between these seemingly different groups is difficult

The semigroup of the trefoil knot

- Generators:
  - a, b, c
- Relations:
  - ab=ca, ba=ac
  - ba=cb, ab=bc
  - ca=bc, ac=cb

- Generators:
  - a, b, c, d, e, f
- Relations:
  - ae=ba, ea=ab
  - be=ec, eb=ce
  - ce=ed, ec=de
  - da=ae, ad=ea
  - ed=df, de=fd
  - fe=eg, ef=ge
  - ge=ea, eg=ae



How easy (algorithmically) is it to get rid of the redundant generators in the presentation on the right and arrive at the 'canonical' presentation on the left?