Languages that require full scanning of words to determine membership

Peter M. Higgins & Suhear Alwan

Department of Mathematical Sciences, University of Essex

NBSAN Norwich April 15th 2014

Compare the five languages:

$$L_1 = aA^* \cup A^*a, \quad L_2 = A^*a, \quad L_3 = aA^*$$

$$L_4 = aA^*b \cup bA^*a, \quad L_5 = (A^2)^*.$$

L_1 has the *Factor Property (FP):*

$$\forall u \in A^* \exists u_1, u_2, u_1', u_2' \in A^* : u_1 u u_2 \in L \,\&\, u_1' u u_2' \in L'$$

L_2 (resp. L_3) has the *Prefix Property (PP)* (resp. *Suffix Property (SP)*):

$$\forall x \in A^* \exists v, v' \in A^* : xv \in L, xv' \in L' \text{(resp. } \exists u, u' \in A^* : ux \in L, u'x \in L').$$

L_4 has both the Prefix and Suffix properties *(Weak Scan property (WS)).*

Compare the five languages:

$$L_1 = aA^* \cup A^*a, \quad L_2 = A^*a, \quad L_3 = aA^*$$

$$L_4 = aA^*b \cup bA^*a, \quad L_5 = (A^2)^*.$$

L_1 has the *Factor Property (FP):*

$$\forall u \in A^* \, \exists \, u_1, u_2, u_1', u_2' \in A^* : u_1 u u_2 \in L \,\&\, u_1' u u_2' \in L'$$

L_2 (resp. L_3) has the *Prefix Property (PP)* (resp. *Suffix Property (SP)*):

$$\forall x \in A^* \, \exists \, v, v' \in A^* : xv \in L, xv' \in L' \, (\text{resp. } \exists u, u' \in A^* : ux \in L, u'x \in L').$$

L_4 has both the Prefix and Suffix properties *(Weak Scan property (WS)).*

Compare the five languages:

$$L_1 = aA^* \cup A^*a, \quad L_2 = A^*a, \quad L_3 = aA^*$$

$$L_4 = aA^*b \cup bA^*a, \quad L_5 = (A^2)^*.$$

L_1 has the *Factor Property (FP):*

$$\forall u \in A^* \, \exists \, u_1, u_2, u_1', u_2' \in A^* : u_1 u u_2 \in L \,\&\, u_1' u u_2' \in L'$$

L_2 (resp. L_3) has the *Prefix Property (PP)* (resp. *Suffix Property (SP)*):

$$\forall x \in A^* \, \exists \, v, v' \in A^* : xv \in L, \, xv' \in L' \, (\text{resp. } \exists u, u' \in A^* : ux \in L, \, u'x \in L').$$

L_4 has both the Prefix and Suffix properties *(Weak Scan property (WS)).*

Compare the five languages:

$$L_1 = aA^* \cup A^*a, \quad L_2 = A^*a, \quad L_3 = aA^*$$

$$L_4 = aA^*b \cup bA^*a, \quad L_5 = (A^2)^*.$$

L_1 has the *Factor Property (FP):*

$$\forall u \in A^* \, \exists \, u_1, u_2, u_1', u_2' \in A^* : u_1 u u_2 \in L \,\&\, u_1' u u_2' \in L'$$

L_2 (resp. L_3) has the *Prefix Property (PP)* (resp. *Suffix Property (SP)*):

$$\forall x \in A^* \, \exists \, v, v' \in A^* : xv \in L, xv' \in L' \,(\text{resp. } \exists u, u' \in A^* : ux \in L, u'x \in L').$$

L_4 has both the Prefix and Suffix properties *(Weak Scan property (WS)).*

A stronger condtion still is the *Full Scan Property (FS):*

$$\forall u, v \in A^* \, \exists x, x' \in A^* : uxv \in \& \, ux'v \in L'.$$

L_5 has the full scan property.

For regular languages, in terms of the minimal automaton $\mathscr{A}(Q, i, T) = \mathscr{A}(L)$ we have:

$$L \in FP \Leftrightarrow \nexists \text{a sink state } q \in Q : Q \cdot z = q \, \forall z \in A^*$$

$$L \in PP \Leftrightarrow |q \cdot A^*| > 1 \, \forall q \in Q; \; L \in SP \Leftrightarrow L^R \in PP$$

$$L \in WS \Leftrightarrow L \in PP \, \& \, L \in SP$$

$$L \in FS \Leftrightarrow Lv^{-1} \in PP \, \forall v \in A^*.$$

A stronger condtion still is the *Full Scan Property (FS):*

$$\forall u, v \in A^* \, \exists x, x' \in A^* : uxv \in \& \, ux'v \in L'.$$

L_5 has the full scan property.
For regular languages, in terms of the minimal automaton $\mathscr{A}(Q, i, T) = \mathscr{A}(L)$ we have:

$$L \in FP \Leftrightarrow \nexists \text{a sink state} \, q \in Q : \, Q \cdot z = q \forall z \in A^*$$

$$L \in PP \Leftrightarrow |q \cdot A^*| > 1 \forall q \in Q; \; L \in SP \Leftrightarrow L^R \in PP$$

$$L \in WS \Leftrightarrow L \in PP \, \& \, L \in SP$$

$$L \in FS \Leftrightarrow Lv^{-1} \in PP \forall v \in A^*.$$

A stronger condtion still is the *Full Scan Property (FS):*

$$\forall u, v \in A^* \, \exists x, x' \in A^* : uxv \in \& \, ux'v \in L'.$$

L_5 has the full scan property.
For regular languages, in terms of the minimal automaton $\mathscr{A}(Q, i, T) = \mathscr{A}(L)$ we have:

$$L \in FP \Leftrightarrow \nexists \text{a sink state}\, q \in Q : \, Q \cdot z = q \forall z \in A^*$$

$$L \in PP \Leftrightarrow |q \cdot A^*| > 1 \forall q \in Q; \, L \in SP \Leftrightarrow L^R \in PP$$

$$L \in WS \Leftrightarrow L \in PP \,\&\, L \in SP$$

$$L \in FS \Leftrightarrow Lv^{-1} \in PP \forall v \in A^*.$$

A stronger condtion still is the *Full Scan Property (FS):*

$$\forall u, v \in A^* \, \exists x, x' \in A^* : \, uxv \in \& \, ux'v \in L'.$$

L_5 has the full scan property.
For regular languages, in terms of the minimal automaton $\mathscr{A}(Q, i, T) = \mathscr{A}(L)$ we have:

$$L \in FP \Leftrightarrow \nexists \text{a sink state} \, q \in Q : \, Q \cdot z = q \forall z \in A^*$$

$$L \in PP \Leftrightarrow |q \cdot A^*| > 1 \forall q \in Q; \, L \in SP \Leftrightarrow L^R \in PP$$

$$L \in WS \Leftrightarrow L \in PP \, \& \, L \in SP$$

$$L \in FS \Leftrightarrow Lv^{-1} \in PP \forall v \in A^*.$$

A stronger condtion still is the *Full Scan Property (FS):*

$$\forall u, v \in A^* \exists x, x' \in A^* : uxv \in \& ux'v \in L'.$$

L_5 has the full scan property.
For regular languages, in terms of the minimal automaton $\mathscr{A}(Q, i, T) = \mathscr{A}(L)$ we have:

$$L \in FP \Leftrightarrow \nexists \text{a sink state}\, q \in Q : Q \cdot z = q \forall z \in A^*$$

$$L \in PP \Leftrightarrow |q \cdot A^*| > 1 \forall q \in Q;\; L \in SP \Leftrightarrow L^R \in PP$$

$$L \in WS \Leftrightarrow L \in PP \,\&\, L \in SP$$

$$L \in FS \Leftrightarrow Lv^{-1} \in PP \forall v \in A^*.$$

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p, q \in A^*) - \eta \text{ saturates } L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.
Conversely, if I is a non-trivial group, then $L \in FS$.

Scanning Conditions in terms of the syntactic monoid

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p,q \in A^*) - \eta \,\text{saturates}\, L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.
Conversely, if I is a non-trivial group, then $L \in FS$.

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p, q \in A^*) - \eta \text{ saturates } L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.
Conversely, if I is a non-trivial group, then $L \in FS$.

Scanning Conditions in terms of the syntactic monoid

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p, q \in A^*) - \eta \, \text{saturates} \, L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.
Conversely, if I is a non-trivial group, then $L \in FS$.

Scanning Conditions in terms of the syntactic monoid

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p, q \in A^*) - \eta \, \text{saturates} \, L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.

Conversely, if I is a non-trivial group, then $L \in FS$.

Scanning Conditions in terms of the syntactic monoid

Let $\overline{(\cdot)} : A^* \to A^*/\eta = M(L)$ denote the natural morphism of A^* onto the syntactic monoid $M(L)$ of language $L \subseteq A^*$, so that

$$\overline{u} = \overline{v} \Leftrightarrow (puq \in L \Leftrightarrow pvq \in L, \forall p, q \in A^*) - \eta \, \text{saturates} \, L.$$

We say a set $X \subseteq M(L)$ is a *bridge* if

$$X\eta^{-1} \cap L \neq \emptyset \,\&\, X\eta^{-1} \cap L' \neq \emptyset.$$

Theorem

Let L be regular and let I be the minimum ideal of $M = M(L)$ Then

(i) $L \in FP$ if and only if the $\mathscr{D}$-class I of M is a bridge;
(ii) $L \in PP$ (resp. SP) if and only if each $\mathscr{R}$-class (resp. $\mathscr{L}$-class) of I is bridge;
(iii) $L \in WS$ if and only if each $\mathscr{R}$-class and each $\mathscr{L}$-class of I is bridge;
(iv) $L \in FS$ if and only if each $\mathscr{H}$-class of I is bridge.
Conversely, if I is a non-trivial group, then $L \in FS$.

Observations & Examples

Examples

(A) $L(f,m,K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0,1,\cdots,m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u,v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is not a bridge, contradiction!

Observations & Examples

Examples

(A) $L(f, m, K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0, 1, \cdots, m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u, v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$,
whence H is not a bridge, contradiction!

Observations & Examples

Examples

(A) $L(f,m,K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0,1,\cdots,m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u,v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is <u>not</u> a bridge, contradiction!

Observations & Examples

Examples

(A) $L(f,m,K)=\{w\in A^*:|w|_f(\text{mod } m)\in K\}$ where $f\in A^*$, $m\geq 2$ and K a proper subset of $\{0,1,\cdots,m-1\}$ is a full scan language.

(B) $L\in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u\in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u,v\in A^*$.
The following languages are not regular:
(i) $\{w\in A^*:|w|_a=|w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L\in FS$ (easy to check) so suppose L were regular and let $\overline{u}\in H$, a maximal subgroup of I.
Take $k\geq 1$ such that $\overline{u}=\overline{u}^{k+1}$: $u^{k+1}\in L'\Rightarrow u\in L'\,\forall\overline{u}\in H$, whence H is not a bridge, contradiction!

Examples

(A) $L(f,m,K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0,1,\cdots,m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u,v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.

Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is not a bridge, contradiction!

Examples

(A) $L(f,m,K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0, 1, \cdots, m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u, v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is not a bridge, contradiction!

Observations & Examples

Examples

(A) $L(f, m, K) = \{w \in A^* : |w|_f (\text{mod } m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0, 1, \cdots, m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u, v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is not a bridge, contradiction!

Observations & Examples

Examples

(A) $L(f,m,K) = \{w \in A^* : |w|_f (\mathrm{mod}\ m) \in K\}$ where $f \in A^*$, $m \geq 2$ and K a proper subset of $\{0,1,\cdots,m-1\}$ is a full scan language.

(B) $L \in FS$ then so is L', L^R, $u^{-1}L$, Lu^{-1} for any $u \in A^*$;
(C) L is full scan if and only if $u^{-1}Lv^{-1}$ is proper for all $u, v \in A^*$.
The following languages are not regular:
(i) $\{w \in A^* : |w|_a = |w|_b\}$; (ii) Primitive words; (iii) Words with borders.
Argument for (ii): $L \in FS$ (easy to check) so suppose L were regular and let $\overline{u} \in H$, a maximal subgroup of I.
Take $k \geq 1$ such that $\overline{u} = \overline{u}^{k+1}$: $u^{k+1} \in L' \Rightarrow u \in L' \forall \overline{u} \in H$, whence H is <u>not</u> a bridge, contradiction!

Chameleon sets

Definition

A set $C \subseteq A^*$ is called a *chameleon set* if $\forall u, v \in A^* \, \exists u', v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Equivalently, each two-sided quotient $u^{-1}Cv^{-1}$ has an empty two-sided quotient $u'^{-1}(u^{-1}Cv^{-1})v'^{-1}$.

Examples

finite sets, complements of ideals.

CP closed under sublanguages, finite unions, reversals, left quotients and right quotients, and so forms a topology on A^*.

Chameleon sets

Definition

A set $C \subseteq A^*$ is called a *chameleon set* if $\forall u, v \in A^* \exists u', v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Equivalently, each two-sided quotient $u^{-1}Cv^{-1}$ has an empty two-sided quotient $u'^{-1}(u^{-1}Cv^{-1})v'^{-1}$.

Examples

finite sets, complements of ideals.

CP closed under sublanguages, finite unions, reversals, left quotients and right quotients, and so forms a topology on A^*.

Chameleon sets

Definition

A set $C \subseteq A^*$ is called a *chameleon set* if $\forall u, v \in A^* \exists u', v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Equivalently, each two-sided quotient $u^{-1}Cv^{-1}$ has an empty two-sided quotient $u'^{-1}(u^{-1}Cv^{-1})v'^{-1}$.

Examples

finite sets, complements of ideals.

CP closed under sublanguages, finite unions, reversals, left quotients and right quotients, and so forms a topology on A^*.

Chameleon sets

Definition

A set $C \subseteq A^*$ is called a *chameleon set* if $\forall u, v \in A^* \exists u', v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Equivalently, each two-sided quotient $u^{-1}Cv^{-1}$ has an empty two-sided quotient $u'^{-1}(u^{-1}Cv^{-1})v'^{-1}$.

Examples

finite sets, complements of ideals.

CP closed under sublanguages, finite unions, reversals, left quotients and right quotients, and so forms a topology on A^*.

Why chameleon?

Theorem

Let L be full scan and C chameleon. Then $L \cup C$ and $L \setminus C$ are full scan.

Proof Let $u, v \in A^*$. Since $C \in CP$ $\exists u'v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Since $L \in FS$ $\exists$ $x, x' \in A^*$ such that $(uu')x(v'v) \in L$ and $(uu')x'(v'v) \in L'$. But then:

$$u(u'xv')v \in L \cup C \text{ and } u(u'x'v')v \in L' \cap C' = (L \cup C)', \text{ thus } L \cup C \in FS.$$

$$L \in FS \Rightarrow L' \cup C \in FS \Rightarrow (L' \cup C)' \in FS \Rightarrow L \cap C' = L \setminus C \in FS. \quad \square$$

Why chameleon?

Theorem

Let L be full scan and C chameleon. Then $L \cup C$ and $L \setminus C$ are full scan.

Proof Let $u, v \in A^*$. Since $C \in CP$ $\exists u'v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Since $L \in FS$ $\exists$ $x, x' \in A^*$ such that $(uu')x(v'v) \in L$ and $(uu')x'(v'v) \in L'$. But then:

$u(u'xv')v \in L \cup C$ and $u(u'x'v')v \in L' \cap C' = (L \cup C)'$, thus $L \cup C \in FS$.

$L \in FS \Rightarrow L' \cup C \in FS \Rightarrow (L' \cup C)' \in FS \Rightarrow L \cap C' = L \setminus C \in FS$. $\square$

Theorem

Let L be full scan and C chameleon. Then $L \cup C$ and $L \setminus C$ are full scan.

Proof Let $u, v \in A^*$. Since $C \in CP$ $\exists u'v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Since $L \in FS$ $\exists$ $x, x' \in A^*$ such that $(uu')x(v'v) \in L$ and $(uu')x'(v'v) \in L'$. But then:

$$u(u'xv')v \in L \cup C \text{ and } u(u'x'v')v \in L' \cap C' = (L \cup C)', \text{ thus } L \cup C \in FS.$$

$$L \in FS \Rightarrow L' \cup C \in FS \Rightarrow (L' \cup C)' \in FS \Rightarrow L \cap C' = L \setminus C \in FS. \quad \square$$

Why chameleon?

Theorem

Let L be full scan and C chameleon. Then $L \cup C$ and $L \setminus C$ are full scan.

Proof Let $u, v \in A^*$. Since $C \in CP$ $\exists u'v' \in A^*$ such that $uu'A^*v'v \cap C = \emptyset$. Since $L \in FS$ $\exists$ $x, x' \in A^*$ such that $(uu')x(v'v) \in L$ and $(uu')x'(v'v) \in L'$. But then:

$$u(u'xv')v \in L \cup C \text{ and } u(u'x'v')v \in L' \cap C' = (L \cup C)', \text{ thus } L \cup C \in FS.$$

$$L \in FS \Rightarrow L' \cup C \in FS \Rightarrow (L' \cup C)' \in FS \Rightarrow L \cap C' = L \setminus C \in FS. \quad \square$$

Theorem

A regular chameleon set has none of the five scanning properties.

In consequence, none of the following languages are regular.

Examples

(A) Language of all palindromes is weak scan and chameleon;

(B) The language of all Lyndon words is chameleon and has the factor property;
(C) The Dyck language (of all meaningful parantheses) is chameleon and has the factor property.

Theorem

A regular chameleon set has none of the five scanning properties.

In consequence, none of the following languages are regular.

Examples

(A) Language of all palindromes is weak scan and chameleon;

(B) The language of all Lyndon words is chameleon and has the factor property;
(C) The Dyck language (of all meaningful parantheses) is chameleon and has the factor property.

Theorem

A regular chameleon set has none of the five scanning properties.

In consequence, none of the following languages are regular.

Examples

(A) Language of all palindromes is weak scan and chameleon;

(B) The language of all Lyndon words is chameleon and has the factor property;

(C) The Dyck language (of all meaningful parantheses) is chameleon and has the factor property.

Theorem

A regular chameleon set has none of the five scanning properties.

In consequence, none of the following languages are regular.

Examples

(A) Language of all palindromes is weak scan and chameleon;

(B) The language of all Lyndon words is chameleon and has the factor property;

(C) The Dyck language (of all meaningful parantheses) is chameleon and has the factor property.

Theorem

A regular chameleon set has none of the five scanning properties.

In consequence, none of the following languages are regular.

Examples

(A) Language of all palindromes is weak scan and chameleon;

(B) The language of all Lyndon words is chameleon and has the factor property;
(C) The Dyck language (of all meaningful parantheses) is chameleon and has the factor property.

Letter scan languages

Take the definition of full scan language and strengthen the condition $uxv \in L$, $ux'v \in L'$ by insisting that $x \in A$. If we take $A = \{a, b\}$ the FSL languages are as follows.

Definitions

Let $E = \{w \in A^* : |w|_a \equiv 0 \pmod 2\}$ and $O = A^* \setminus E$. Let $E_n = E \cap A^n$, $O_n = O \cap A^n$. For any $L \subseteq A^*$ let $L_n = L \cap A^n$.

Theorem

L is FSL if and only if $L_n \in \{E_n, O_n\} \forall n \geq 0\}$.

There is then a one-to-one correspondence between FSL languages L and real numbers s_L in the interval $[0, 2]$: the initial digit determines the presence or absence of ε, the nth digit is 0 if and only if $L_n = E_n$.

Theorem

Let $L \in$FSL. Then L is regular if and only if $s_L \in \mathbb{Q}$.

Letter scan languages

Take the definition of full scan language and strengthen the condition $uxv \in L$, $ux'v \in L'$ by insisting that $x \in A$. If we take $A = \{a, b\}$ the FSL languages are as follows.

Definitions

Let $E = \{w \in A^* : |w|_a \equiv 0 (\mathrm{mod}\ 2)\}$ and $O = A^* \setminus E$. Let $E_n = E \cap A^n$, $O_n = O \cap A^n$. For any $L \subseteq A^*$ let $L_n = L \cap A^n$.

Theorem

L is FSL if and only if $L_n \in \{E_n, O_n\} \forall n \geq 0\}$.

There is then a one-to-one correspondence between FSL languages L and real numbers s_L in the interval $[0, 2]$: the initial digit determines the presence or absence of ε, the nth digit is 0 if and only if $L_n = E_n$.

Theorem

Let $L \in$FSL. Then L is regular if and only if $s_L \in \mathbb{Q}$.

Letter scan languages

Take the definition of full scan language and strengthen the condition $uxv \in L$, $ux'v \in L'$ by insisting that $x \in A$. If we take $A = \{a, b\}$ the FSL languages are as follows.

Definitions

Let $E = \{w \in A^* : |w|_a \equiv 0 \pmod 2\}$ and $O = A^* \setminus E$. Let $E_n = E \cap A^n$, $O_n = O \cap A^n$. For any $L \subseteq A^*$ let $L_n = L \cap A^n$.

Theorem

L is FSL if and only if $L_n \in \{E_n, O_n\} \forall n \geq 0\}$.

There is then a one-to-one correspondence between FSL languages L and real numbers s_L in the interval $[0, 2]$: the initial digit determines the presence or absence of ε, the nth digit is 0 if and only if $L_n = E_n$.

Theorem

Let $L \in$FSL. Then L is regular if and only if $s_L \in \mathbb{Q}$.

Take the definition of full scan language and strengthen the condition $uxv \in L$, $ux'v \in L'$ by insisting that $x \in A$. If we take $A = \{a, b\}$ the FSL languages are as follows.

Definitions

Let $E = \{w \in A^* : |w|_a \equiv 0 \pmod 2\}$ and $O = A^* \setminus E$. Let $E_n = E \cap A^n$, $O_n = O \cap A^n$. For any $L \subseteq A^*$ let $L_n = L \cap A^n$.

Theorem

L is FSL if and only if $L_n \in \{E_n, O_n\} \forall n \geq 0\}$.

There is then a one-to-one correspondence between FSL languages L and real numbers s_L in the interval $[0, 2]$: the initial digit determines the presence or absence of ε, the nth digit is 0 if and only if $L_n = E_n$.

Theorem

Let $L \in$FSL. Then L is regular if and only if $s_L \in \mathbb{Q}$.

Letter scan languages

Take the definition of full scan language and strengthen the condition $uxv \in L$, $ux'v \in L'$ by insisting that $x \in A$. If we take $A = \{a, b\}$ the FSL languages are as follows.

Definitions

Let $E = \{w \in A^* : |w|_a \equiv 0 \pmod 2\}$ and $O = A^* \setminus E$. Let $E_n = E \cap A^n$, $O_n = O \cap A^n$. For any $L \subseteq A^*$ let $L_n = L \cap A^n$.

Theorem

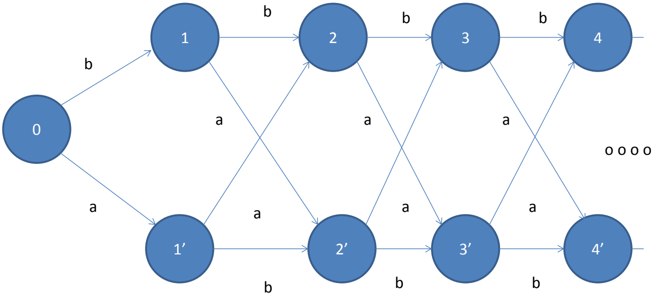
L is FSL if and only if $L_n \in \{E_n, O_n\} \forall n \geq 0\}$.

There is then a one-to-one correspondence between FSL languages L and real numbers s_L in the interval $[0, 2]$: the initial digit determines the presence or absence of ε, the nth digit is 0 if and only if $L_n = E_n$.

Theorem

Let $L \in$FSL. Then L is regular if and only if $s_L \in \mathbb{Q}$.

Universal FSL Automaton



Minimal automaton in rational case

The universal automaton $\mathscr{U}$ will recognize a given FSL language L by putting n or $n' \in T$ according as $L_n = E_n$ or $L_n = O_n$. In effect we just read s_L into $\mathscr{U}$.

Let $s_L = e_0 \cdot e_1 e_2 \cdots$. If $s_L \in \mathbb{Q}$ with $e_k = e_{k+n}$ for some minimum k and n, then we may identify the pairs of states $(k, k+n)$ and $(k', (k+n)')$. The resulting finite automaton $\mathscr{A}(L)$ has the form of a cylinder with a trailing tape that leads to a point (0): and $\mathscr{A}(L)$ is the minimal automaton of L EXCEPT if s_L has the form:

$$s_L = \frac{1}{2^k}\left(n + \frac{t}{1+2^r}\right), 0 \le k, 0 \le n \le 2^k - 1, 1 \le r, 1 \le t \le 2^r.$$

Minimal automaton in rational case

The universal automaton $\mathscr{U}$ will recognize a given FSL language L by putting n or $n' \in T$ according as $L_n = E_n$ or $L_n = O_n$. In effect we just read s_L into $\mathscr{U}$.

Let $s_L = e_0 \cdot e_1 e_2 \cdots$. If $s_L \in \mathbb{Q}$ with $e_k = e_{k+n}$ for some minimum k and n, then we may identify the pairs of states $(k, k+n)$ and $(k', (k+n)')$. The resulting finite automaton $\mathscr{A}(L)$ has the form of a cylinder with a trailing tape that leads to a point (0):

and $\mathscr{A}(L)$ is the minimal automaton of L EXCEPT if s_L has the form:

$$s_L = \frac{1}{2^k}\left(n + \frac{t}{1+2^r}\right), 0 \leq k, 0 \leq n \leq 2^k - 1, 1 \leq r, 1 \leq t \leq 2^r.$$

Minimal automaton in rational case

The universal automaton $\mathscr{U}$ will recognize a given FSL language L by putting n or $n' \in T$ according as $L_n = E_n$ or $L_n = O_n$. In effect we just read s_L into $\mathscr{U}$.

Let $s_L = e_0 \cdot e_1 e_2 \cdots$. If $s_L \in \mathbb{Q}$ with $e_k = e_{k+n}$ for some minimum k and n, then we may identify the pairs of states $(k, k+n)$ and $(k', (k+n)')$. The resulting finite automaton $\mathscr{A}(L)$ has the form of a cylinder with a trailing tape that leads to a point (0): and $\mathscr{A}(L)$ is the minimal automaton of L EXCEPT if s_L has the form:

$$s_L = \frac{1}{2^k}\left(n + \frac{t}{1+2^r}\right), 0 \leq k, 0 \leq n \leq 2^k - 1, 1 \leq r, 1 \leq t \leq 2^r.$$

The universal automaton $\mathscr{U}$ will recognize a given FSL language L by putting n or $n' \in T$ according as $L_n = E_n$ or $L_n = O_n$. In effect we just read s_L into $\mathscr{U}$.

Let $s_L = e_0 \cdot e_1 e_2 \cdots$. If $s_L \in \mathbb{Q}$ with $e_k = e_{k+n}$ for some minimum k and n, then we may identify the pairs of states $(k, k+n)$ and $(k', (k+n)')$. The resulting finite automaton $\mathscr{A}(L)$ has the form of a cylinder with a trailing tape that leads to a point (0):

and $\mathscr{A}(L)$ is the minimal automaton of L EXCEPT if s_L has the form:

$$s_L = \frac{1}{2^k}\left(n + \frac{t}{1+2^r}\right), 0 \leq k, 0 \leq n \leq 2^k - 1, 1 \leq r, 1 \leq t \leq 2^r.$$

Cylinder versus Mobius strip

This special case is where the recurring part of s_L has the form $z\overline{z}$ where $\overline{z}$ is defined by $z+\overline{z}=11\cdots1$ (with $2r$ 1's), so that $\overline{z}$ is the obtained from z by interchanging the symbols 0 and 1 throughout.

In this case the cylinder of circumference $2r$ may be replaced by a Mobius strip of edge length $2r$:

We may identify the pairs of states $(k+r,k')$ and $((k+r)',k)$, the resulting half-twist giving the form of a Mobius strip.

Cylinder versus Mobius strip

This special case is where the recurring part of s_L has the form $z\overline{z}$ where $\overline{z}$ is defined by $z + \overline{z} = 11\cdots1$ (with $2r$ 1's), so that $\overline{z}$ is the obtained from z by interchanging the symbols 0 and 1 throughout. In this case the cylinder of circumference $2r$ may be replaced by a Mobius strip of edge length $2r$:

We may identify the pairs of states $(k+r, k')$ and $((k+r)', k)$, the resulting half-twist giving the form of a Mobius strip.

Cylinder versus Mobius strip

This special case is where the recurring part of s_L has the form $z\overline{z}$ where $\overline{z}$ is defined by $z + \overline{z} = 11 \cdots 1$ (with $2r$ 1's), so that $\overline{z}$ is the obtained from z by interchanging the symbols 0 and 1 throughout. In this case the cylinder of circumference $2r$ may be replaced by a Mobius strip of edge length $2r$:
We may identify the pairs of states $(k+r, k')$ and $((k+r)', k)$, the resulting half-twist giving the form of a Mobius strip.