# Constraint Satisfaction Problems with Tree Duality

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CSP:

Input: relational structures $G, H$ (over same signature $\sigma$)

Output: Yes if there is a homomorphism $G \longrightarrow H$

No otherwise

("Everything" is finite.)

# Computational Complexity

P = problems that admit a polynomial-time algorithm
NP = problems with a polynomial-size certificate for YES, which can be checked in polynomial time

NP-hard = every problem in NP reducible to it
NP-complete = NP-hard $\cap$ NP

Examples:

$P \ni$ 2-colouring of graphs

3-colouring of graphs is NP-complete

CSP is NP-complete

## CSP is NP-complete

→ Exponential algorithms

→ Poly-time algorithms that don't always work

→ Restrict the constraints that are allowed (and classify complexity)

↓

CSP(H) : H is fixed

Input: G

Question: $G \to H$ ?

We've seen $H = K_2 \quad K_3$

CSP(H) : H is fixed

Input: G

Question: $G \to H$ ?

More examples:

- one ternary relation; dom $H = \{0,1\}$

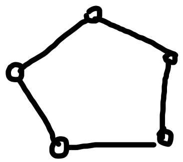
  $x + y + z = 1$ (in GF(2))

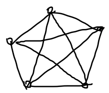
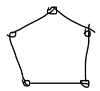
- dom $H = \{0,1\}$; four ternary relations to express clauses such as $(x \vee \neg y \vee \neg z)$ 3-SAT

CSP (H):

→ Sometimes P, sometimes NP-cp'te, sometimes... ???

→ Hell, Nešetřil, 1990: For symmetric graphs:
- P if H is bipartite or has a loop
- NP-complete otherwise

Example: $H = C_5 =$ 

Observe: $G \to$  $\iff G^{1/3} \to$ 

$$G \to H^3 \iff G^{1/3} \to H$$

... adjoint functors

CSP (H):

→ Sometimes P, sometimes NP-cp'te, sometimes... ???

→ Hell, Nešetřil, 1990: For symmetric graphs:
- P if H is bipartite or has a loop
- NP-complete otherwise

→ Feder, Vardi, 1998:
- conjectured dichotomy
- studied "width-1" problems
- lots of other things

→ Hell, Nešetřil, Zhu, 1996:
- "tree duality" problems
- "bounded treewidth duality"

$\}$ for digraphs

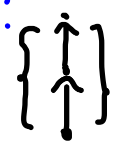
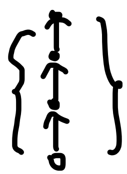
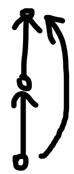
H has **tree duality** if
whenever $G \not\to H$,
then there is a $\sigma$-tree $F$ s.t.
$F \to G$ and $F \not\to H$.

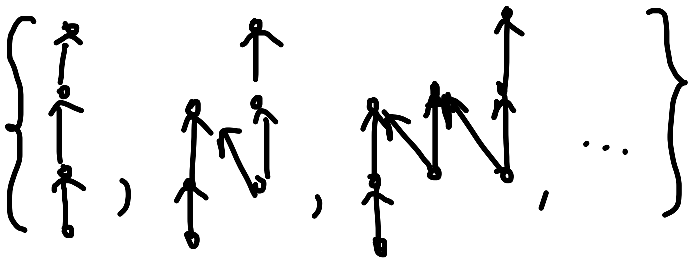
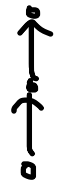
$\Updownarrow$

there exists $\mathcal{F}$ consisting of $\sigma$-trees only, s.t.

- whenever $G \not\to H$, then $\exists F \in \mathcal{F}$, $F \to G$
- $F \in \mathcal{F} \Rightarrow F \not\to H$

$\mathcal{F}$ is a **complete set of obstructions** for $H$

Examples:  for $H =$     for $H =$ 

 for $H =$ 

# Arc consistency

$L : \mathrm{dom}\, G \longrightarrow 2^{\mathrm{dom}\, H}$ is consistent with an arc/tuple $(x_1, \dots, x_k) \in R^G$ $(R \in \sigma)$ if

$$\forall i \ \ \forall y_i \in L(x_i) \ \exists y_1, y_2, \dots, y_{i-1}, y_{i+1}, y_{i+2}, \dots, y_k \in \mathrm{dom}\, H,$$
each $y_j \in L(x_j)$,
s.t. $(y_1, \dots, y_k) \in R^H$.

**Algorithm** for CSP(H)

Input : G

1. Initialise $L(x) := \mathrm{dom}\, H \quad \forall x \in \mathrm{dom}\, G$
2. While $\forall x,\ L(x) \neq \emptyset$:
   - If $L$ is inconsistent with some $(x_1, \dots, x_k) \in R^G$, remove corresponding $y_i$ from $L(x_i)$.
   - If $L$ is consistent with all tuples of $G$, stop.

**Algorithm** for CSP(H)

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① Initialise $L(x) := \mathrm{dom}\, H \quad \forall x \in \mathrm{dom}\, G$

② While $\forall x,\ L(x) \neq \emptyset$:

- If $L$ is inconsistent with some $(x_1, \dots, x_k) \in R^G$, remove corresponding $y_i$ from $L(x_i)$.
- If $L$ is consistent with all tuples of $G$, stop.

- If some $L(x) = \emptyset$, then $G \not\to H$.
- If each $|L(x)| = 1$, then $G \to H$.
- If $G$ is a tree, non-empty $L$ gives a homomorphism $G \to H$; if $F$ is a tree and $g: F \to G$, then $L(g(z))$ gives a hom. $F \to H$. Hence the algorithm is correct for CSP(H) with tree duality!
- Without tree duality, non-empty lists do not guarantee $G \to H$.

But how do we find out if $H$ has tree duality?

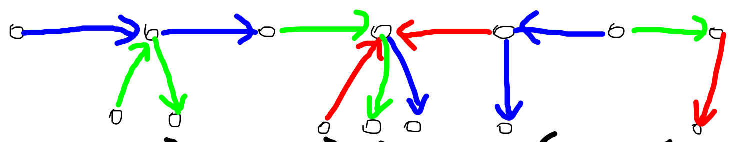
Feder, Vardi: power structure $\mathcal{U}(H)$:

- $\mathrm{dom}\, \mathcal{U}(H) = 2^{\mathrm{dom}\, H} \setminus \{\emptyset\}$
- $(A_1, \dots, A_k) \in R^{\mathcal{U}(H)}$ if $\forall i \; \forall x_i \in A_i \quad \exists x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k$ each $x_j \in A_j$, s.t. $(x_1, \dots, x_k) \in R^H$.

$$\boxed{H \text{ has tree duality} \quad \text{iff} \quad \mathcal{U}(H) \to H}$$

# Which $\mathcal{F}$'s are complete sets of obstructions for CSP(H)?

Simplified setup: Consider $\sigma$ containing only binary relation symbols; assume all elements of $\mathcal{F}$ are caterpillars:



$= F$

associate a word with $F$: $\vec{B}\, g^i g^o\, \vec{B}\, \vec{G}\, r^i g^o b^o\, \overleftarrow{R}\, b^o\, \overleftarrow{B}\, \vec{G}\, r^o$ $\longrightarrow$ language $\mathcal{L}(\mathcal{F})$

P.L. Erdős, C. Tardif, G. Tardos (2013):

1. If $\mathcal{L}(\mathcal{F})$ is regular, then $\mathcal{F}$ is a complete set of obstructions for some (finite) H.
2. If $\mathcal{F}$ is a c.s.o. for H, then $\mathcal{L}(\uparrow\mathcal{F})$ is a regular language.

Extends to any $\sigma$ and non-caterpillars.

$\mathcal{F}$ is a c.s.o. for some finite $H$

$\mathrm{Forb}(\mathcal{F}) = \mathrm{CSP}(H)$

$\Updownarrow$ Erdős, Pálvölgyi, Tardif, Tardos

$\mathcal{F}$ is a regular set of $\sigma$-trees

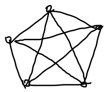
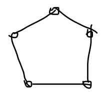
$\Updownarrow$ Hubička, Nešetřil

There is an infinite universal "limit" structure $L$, $\mathrm{CSP}(H) = \mathrm{Age}(L)$, and $L$ is $\omega$-categorical.

$\Updownarrow$ J. F.

There is a finite signature $\tau \supseteq \sigma$, extending $\sigma$ by unary relation symbols, and a universal $\tau$-structure $L^*$, s.t. $L$ is the $\sigma$-reduct of $L^*$ and $L^*$ is a Ramsey structure ($\equiv \mathrm{Aut}(L^*)$ is extremely amenable).

# Adjoint Functors

$G \to$  $\Longleftrightarrow$ $G^{1/3} \to$ 

$$G \to H^3 \Longleftrightarrow G^{1/3} \to H$$

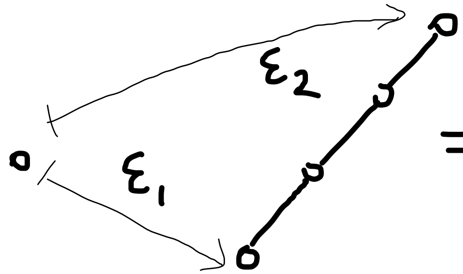
A. Pultr, 1970: The right adjoints in the category of $\sigma$-structures are given by: $A$, $B_R$ for each $R \in \sigma$ of arity $k$,
homs $\varepsilon_i : A \to B_R$ for $i = 1, 2, \ldots, k$.

$H \mapsto \Gamma(H)$: $\operatorname{dom} \Gamma(H) = \operatorname{Hom}(A, H)$

for $R \in \sigma$: $R^{\Gamma(H)} = \operatorname{Hom}(B_R, H)$,

$g : B_R \to H$

$g$ is the tuple $(g \circ \varepsilon_1, g \circ \varepsilon_2, \ldots, g \circ \varepsilon_k)$.

Example above: $A =$ 

$= B$

Let $\Lambda, \Gamma$ be functors $Rel(\sigma) \to Rel(\sigma)$; $\Lambda \dashv \Gamma$.

then $\forall G, H$: $\Lambda(G) \to H \iff G \longrightarrow \Gamma(H)$

$\Lambda(G)$ can be constructed in polynomial time

<u>Therefore</u>: • If $CSP(\Gamma(H))$ is NP-complete, then so is $CSP(H)$.

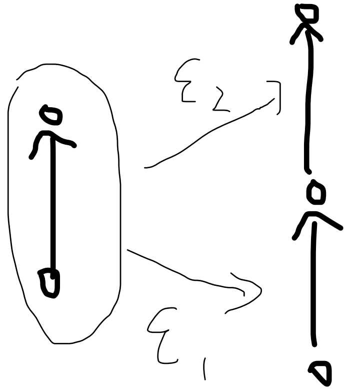
• If $CSP(H)$ is poly-time, then so is $CSP(\Gamma(H))$.

<u>But also</u>: • If $CSP(H)$ has tree duality, then so does $CSP(\Gamma(H))$.

(J.F., C. Tardif, 2009)

**But also**: • If CSP(H) has tree duality, then so does CSP($\Gamma$(H)).
(J.F., C. Tardif, 2009)

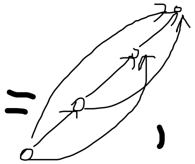
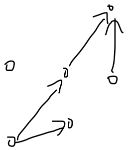
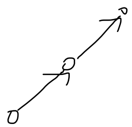
**Example**: arc graph $\delta$

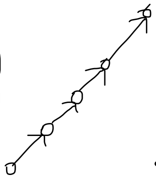
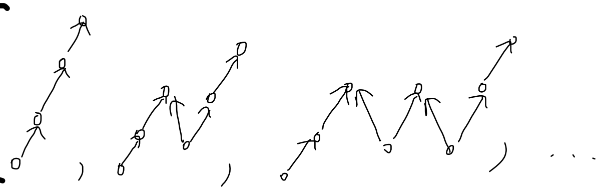
$A =$ 

$= B$

In this case we know a description of the obstructions: Sproink.

> • If $\mathcal{F}$ is a complete set of tree obstructions for H, then Sproink($\mathcal{F}$) is a c.s.o. for $\delta$(H).

e.g. $H =$  , $\delta(H) =$  $\approx$ 

$\mathcal{F} = \{$  $\}$, Sproink($\mathcal{F}$) $= \{$  $\}$

# Finite Duality

$H$ has finite duality if it admits a finite c.s.o.

J. Nešetřil, C. Tardif, 2000:

- If $H$ admits a finite c.s.o., then it admits a finite c.s.o. of trees. (i.e., finite duality $\Rightarrow$ tree duality)
- Any finite set $\mathcal{F}$ of trees is a c.s.o. for some dual $H = D(\mathcal{F})$.

A. Atserias 2005 / B. Rossman 2005:

- CSP($H$) is first-order definable
  $\Leftrightarrow$ $H$ has finite duality.

Digraphs: "$\longrightarrow$" ($\equiv$ existence of a homomorphism) is a pre-order on the set of all digraphs.

$\mathcal{F}$... finite c.s. of tree obstructions for $H$

$\Rightarrow$ $\mathcal{F} \cup \{H\}$ is a finite maximal antichain (or $\mathcal{F}$ is)

CSP$(H_1, \ldots, H_n)$: Does $G$ admit a homomorphism to any of $H_1, \ldots, H_n$?

If CSP$(H_1, \ldots, H_n)$ admits a finite c.s.o. $\mathcal{F}$, then all elements of $\mathcal{F}$ are forests.

J.F., J. Nešetřil, C. Tardif, 2008:
The finite maximal antichains in "$\rightarrow$" are exactly the sets $\mathcal{F} \cup \{H_i : 1 \leq i \leq n;\ \forall F \in \mathcal{F},\ H_i \not\rightarrow F\}$ where $\mathcal{F}$ is a finite c.s.o. for CSP$(H_1, \ldots, H_n)$.

# To summarise:

CSP(H) with tree duality is interesting (to me) because it reaches out to many various areas:

- Ramsey theory
- regular languages (+ Datalog)
- logic (first-order definability)
- categories (adjoints)
- universal algebra (which I didn't talk about)