CSPs and dualities

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Constraint Satisfaction Problem

CSP

Given two finite relational structures
$\mathcal{A} = (A; R_1{}^{\mathcal{A}}, \ldots, R_m{}^{\mathcal{A}})$ and $\mathcal{B} = (B; R_1{}^{\mathcal{B}}, \ldots, R_m{}^{\mathcal{B}})$
is there a homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$?

Example

A graph is a relation structure with exactly one binary relation: the edge relation.

Can one graph be mapped homomorphically to another graph?

Example

The domain $B = \{-1, 0, 1\}$ with ternary relations

$$R_1 = \{(x, y, z) \in B^3 : x + y + z \geq 1\}$$

and

$$R_2 = \{(-x, -y, -z) : (x, y, z) \in R_1\}$$

forms a relational structure $\mathcal{B} = (B; R_1, R_2)$.

$(1, 0, 0), (1, 1, -1) \in R_1$ and $(1, 0, -1) \notin R_1$
$(-1, 0, 0) \in R_2$ and $(1, 0, -1) \notin R_1$, actually $R_1 \cap R_2 = \emptyset$

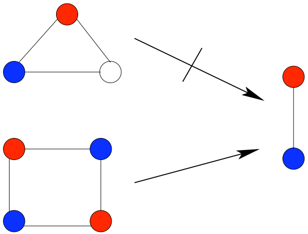
Non-uniform CSP

We fix a target structure $\mathcal{B}$ and ask which structures (with the same signature) admit a homomorphism to $\mathcal{B}$

$CSP(\mathcal{B}) = \{\mathcal{A} : \mathcal{A} \longrightarrow \mathcal{B}\}$

Example

The 2-colourability problem is equivalent to $CSP(K_2)$.



Complexity of CSP

Problem: Classify $CSP(\mathcal{B})$ wrt computational complexity.

Dichotomy Conjecture (Feder/Vardi '98)

For each $\mathcal{B}$, the problem $CSP(\mathcal{B})$ is either tractable (i.e., in **P***) or* **NP***-complete.*

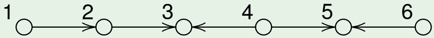
How can this be done? We like algebra

Polymorphisms

A polymorphism f of a structure $\mathcal{B}$ is an n-ary operation in B that is a homomorphism $f : \mathcal{B}^n \longrightarrow \mathcal{B}$.

Example

Oriented paths have polymorphisms $\min(x_1, \ldots, x_n)$ for every $n \geq 1$.



An n-ary operation f is

- a projection on coordinate i if $f(x_1, \ldots, x_n) = x_i$
- idempotent if $f(x, \ldots, x) = x$,
- symmetric if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for any permutation π of $\{1, \ldots, n\}$,
- totally symmetric (TS) if $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ whenever $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$,
- near-unanimity (NU) if $f(x, y, \ldots, y) = f(y, x, y, \ldots, y) = \cdots = f(y, \ldots, y, x) = y$

Example

Meet on a semilattice is a TSI operation. It can be defined of any arity we want.

Example

On $B = \{-1, 0, 1\}$ we define and n-ary operation as follows

$$s_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 + \cdots + x_n = 0 \\ -1 & \text{if } x_1 + \cdots + x_n < 0 \\ 1 & \text{if } x_1 + \cdots + x_n > 0 \end{cases}$$

For any n this operation is symmetric and idempotent.

Corollary (Bulatov, Jeavons; Willard)

If $\mathcal{B}_1$ and $\mathcal{B}_2$ support the same strong Maltsev conditions then CSP($\mathcal{B}_1$) and CSP($\mathcal{B}_2$) are equivalent.

polymorphisms control the complexity of the CSP

A strong Maltsev condition is any finite set of identities
Generally, a strong Maltsev condition may involve many functions and/or superpositions.

Algebraic Conjecture (FV'98, Bulatov, Jeavons, Krokhin '05)

For each core structure $\mathcal{B}$

- *either all polymorphisms of $\mathcal{B}^c$ are projections, and CSP($\mathcal{B}$) is* **NP***-complete,*
- *or else $\mathcal{B}^c$ has a Taylor polymorphism of some arity and CSP($\mathcal{B}$) is tractable.*

A structure is a core if every endomorphism is an automorphism.

$\mathcal{B}^c$ is the structure $\mathcal{B}$ together with all constants, i.e. unary relations $\{a\}$ for every a in the domain. We only need to consider idempotent polymorphisms, i.e. $f(x, \ldots, x) = x$

Theorem

For any structure $\mathcal{B}$, tfae:

1. $\mathcal{B}^c$ *has a Taylor polymorphism*
2. $\mathcal{B}^c$ *has a weak near-unanimity polymorphism [Maroti,McKenzie'06]*

$$f(y,x,...,x,x) = f(x,y,...,x,x) = ... = f(x,x,...,x,y)$$

3. $\mathcal{B}^c$ *has a cyclic polymorphism [Barto,Kozik'11]*

$$f(x_1,x_2,x_3,...,x_n) = f(x_2,x_3,...,x_n,x_1)$$

4. $\mathcal{B}^c$ *has a Siggers polymorphism [Siggers'09,KMM'09]*

$$f(a,r,e,a) = f(r,a,r,e)$$

Theorem (Barto, Kozik, Niven' 09/10)

Tfae (roughly)

- $\mathcal{B}$ *has a cyclic polymorphism;*
- $\mathcal{B}$ *has a lot of cyclic polymorphisms of arities greater than the size of the domain* $|B|$.

How does a lot differ from all?

What does "of all arities" do?

Duality land

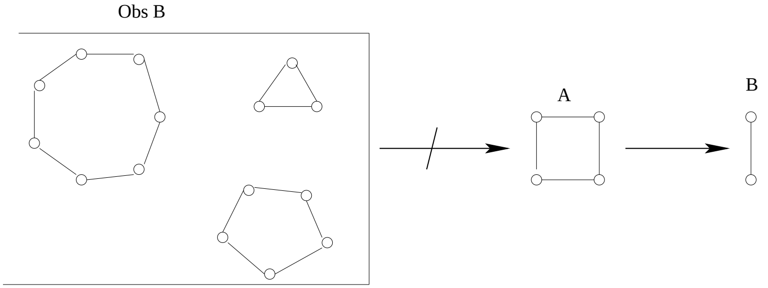
The idea is to justify the existence of a homomorphism by the non-existence of other homomorphisms.

If all structures $\mathcal{A} \not\longrightarrow \mathcal{B}$ can be characterized in uniform way then we can obtain information about the complexity of *CSP*($\mathcal{B}$).

Obstruction sets

An obstruction set for a structure $\mathcal{B}$ is a class $\mathcal{O}_{\mathcal{B}}$ of structures such that, for all structures $\mathcal{A}$

$$\mathcal{A} \mapsto \mathcal{B} \text{ iff } \mathcal{A}' \not\mapsto \mathcal{A} \text{ for all } \mathcal{A}' \in \mathcal{O}_{\mathcal{B}}.$$



Example

If $\mathcal{B}$ is a bipartite graph then $\mathcal{O}_{\mathcal{B}}$ can be chosen to consist of all

Dualities

A structure $\mathcal{B}$ has "nice" duality if $\mathcal{O}_{\mathcal{B}}$ can be chosen to be "simple":

Duality	$\mathcal{O}_{\mathcal{B}}$	Example $\mathcal{B}$
finite	finite	transitive tournament
path	consisting of "paths"	oriented path
...	...	...
tree	consisting of "trees"	Horn 3-SAT $x \wedge y \to z,\ \overline{x} \vee \overline{y} \vee \overline{z}, x$

Trees

The incidence multigraph of $\mathcal{A}$ is a bipartite multigraph with vertices

- all elements of A and;
- all pairs (blocks) $(R, (a_1, \ldots, a_n))$, with R a relation of $\mathcal{A}$ and $(a_1, \ldots, a_n)$ a tuple in R.

$a \in A$ is connected to $(R, (a_1, \ldots, a_n))$ iff $a = a_i$.

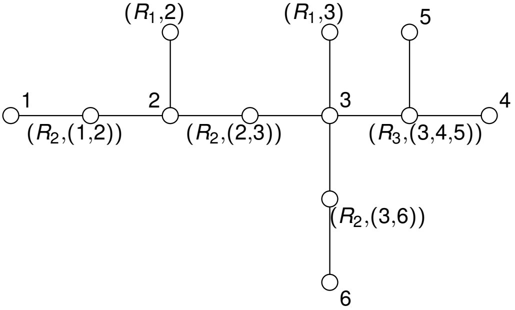
A structure $\mathcal{A}$ is a τ-tree, or just tree, if its incidence multigraph is a tree, i.e. has no cycles or multiple edges.

Example

If τ is the signature of digraphs then τ-trees are exactly the oriented trees.

Example

The structure $\mathcal{A}$ with domain $\{1, \ldots, 6\}$ and relations $R_1 = \{2, 3\}, R_2 = \{(1, 2), (2, 3), (3, 6)\}, R_3 = \{(3, 4, 5)\}$ is a tree.



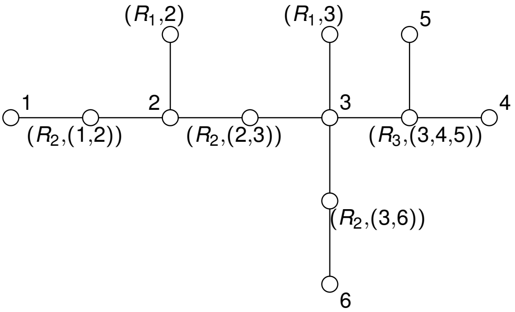
Some dualities

1. $\mathcal{B}$ has finite duality iff $CSP(\mathcal{B})$ is FO-definable iff $CSP(\mathcal{B})$ is in non-uniform AC^0 (Larose, Loten, Tardif'07; Libkin'04)
2. if $\mathcal{B}$ has bounded pathwidth duality then $CSP(\mathcal{B})$ is in NL (Dalmau'05)
3. $\mathcal{B}$ has bounded treewidth duality iff it has weak-NU polymorphisms of all but finitely many arities (Barto, Kozik '09), then $CSP(\mathcal{B})$ is in P
4. $\mathcal{B}$ has tree duality iff it has TSIs of all arities (Dalmau, Pearson '99)

Caterpillars

A structure $\mathcal{A}$ is a τ-path if *Inc*($\mathcal{A}$) is a tree with two "end" blocks.

$\mathcal{A}$ is a τ-caterpillar if it is a τ-path with extra block legs.



$\mathcal{A} = (\{1,\ldots,6\};\ \ \{2,3\},\ \ \{(1,2),(2,3),(3,6)\},\ \ \{(3,4,5)\}$ is a caterpillar.

More polymorphisms

A (mn)-ary operation f is *m-block symmetric* if
$f(S_1, \ldots, S_n) = f(T_1, \ldots, T_n)$
whenever $\{S_1, \ldots, S_n\} = \{T_1, \ldots, T_n\}$, with $S_i = \{x_{i1}, \ldots, x_{im}\}$.

f is an *m-ABS operation* if it is m-block symmetric and it satisfies the absorptive rule
$f(S_1, S_2, S_3, \ldots, S_n) = f(S_2, S_2, S_3, \ldots, S_n)$ whenever $S_2 \subseteq S_1$.

Example

For a fixed linear order the operation
$\min(\max(x_{11}, \ldots, x_{1m}), \ldots, \max(x_{n1}, \ldots, x_{nm}))$ is an m-ABS operation.

Like block cyphers with extra absorption!

Caterpillar duality

m-ABS operations generalize
$(x_1 \sqcap \ldots \sqcap x_m) \sqcup \ldots \sqcup (x_{jm+1} \sqcap \ldots \sqcap x_{(j+1)m})$.

Theorem (C., Dalmau, Krokhin)

Tfae

1. $\mathcal{B}$ *has caterpillar duality;*
2. *co-CSP($\mathcal{B}$) is definable by a linear monadic Datalog program with at most one EDB per rule;*
3. $\mathcal{B}$ *has m-ABS polymorphisms of arity mn, for all* $m, n \geq 1$*;*
4. $\mathcal{B}$ *is homomorphically equivalent to a structure* $\mathcal{A}$ *with polymorphisms* $x \sqcap y$ *and* $x \sqcup y$ *for some distributive lattice* $(A, \sqcup, \sqcap)$*;*
5. $\mathcal{B}$ *is homomorphically equivalent to a structure* $\mathcal{A}$ *with polymorphisms* $x \sqcap y$ *and* $x \sqcup y$ *for some lattice* $(A, \sqcup, \sqcap)$*.*

Caterpillars and regular languages

Characterizing obstruction sets: given a family $\mathcal{O}$ is there a structure $\mathcal{B}$ s.t. $\mathcal{O}$ is an obstruction set for $\mathcal{B}$.

Theorem (Nesestril, Tardif '00)

If a structure has finite duality then it has a finite obstruction set consisting of trees.

Theorem (Erdős, Tardif, Tardos '12)

Let $\mathcal{L}$ be a language, $\mathcal{O}$ the family of caterpillars described by $\mathcal{L}$. Then $\mathcal{O}$ is an obstruction set for a structure $\mathcal{A}$ iff $\mathcal{L}^+$ is regular.

The family of caterpillar obstructions for a structure is described by a regular language.

Example (Kun)

$B = \{-1, 0, 1\}$ with ternary relations

$$R_1 = \{(x, y, z) \in B^3 : x + y + z \geq 1\}$$

and

$$R_2 = \{(-x, -y, -z) : (x, y, z) \in R_1\}$$

is preserved by symmetric operations

$$s_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 + \cdots + x_n = 0 \\ -1 & \text{if } x_1 + \cdots + x_n < 0 \\ 1 & \text{if } x_1 + \cdots + x_n > 0 \end{cases}$$

but not by TSI of arity 3.

algebra rocks <3

Questions

So, tree duality is characterised by TSIs of all arities.

What duality do we get from SIs of all arities?

What about cyclic of all arities?

Symmetric does not imply cyclic

Theorem (C., Krokhin)

If an algebra has term operations of arities 2 and 3 then it also has symmetric term operations of arities up to 4.

Theorem (C., Krokhin)

There exists a structure (domain size 21) preserved by cyclic polymorphisms of all arities, but no symmetric polymorphism of arity 5.

Given by the group A_5.

Proposition (Barto et al.)

Let $\mathcal{A}$ be a finite algebra.

- *Either $\mathcal{A}$ has cyclic term operations of all arities,*
- *or else there is a finite algebra $\mathcal{B}$ in $\mathcal{V}(\mathcal{A})$ with a fixed-point-free automorphism.*

Theorem (C., Krokhin)

Let $\mathcal{A}$ be a finite algebra.

- *Either $\mathcal{A}$ has symmetric term operations of all arities,*
- *or else there is a finite algebra $\mathcal{B}$ in $\mathcal{V}(\mathcal{A})$ that has two automorphisms without a common fixed point. Furthermore, one of the automorphisms can be chosen to have order two.*

Open questions

- As it stands having cyclic operations of al arities but no symmetric operations of all arities is a property expressible in the variety, can it be expressed just in HS? I.e. without using finite products.
- Do these properties collapse with any natural added assumptions?
- What dualities do we have here?