Subsemigroup growth of finitely generated free semigroups

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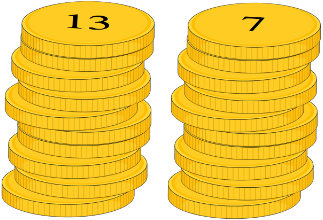
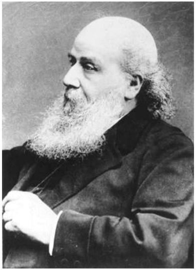
Sylvester's Problem





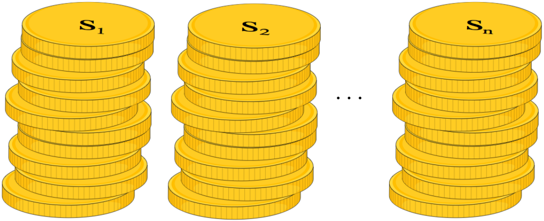
Sylvester stated and solved the following problem:

> *Let s_1 and s_2 be two relatively prime natural numbers. Determine the largest integer g which can not be written as a linear combination $n_1 s_1 + n_2 s_2$ where n_1 and n_2 are non-negative integers.*
>
> *(Mathematical Questions with their Solutions, Educational Times vol. 41, 1884)*

Answer: $s_1 s_2 - s_1 - s_2$.

For example, when $s_1 = 13$, $s_2 = 7$, we get 71.

Frobenius problem



Frobenius proposed a generalisation of Sylvester's problem:

Frobenius problem

Let $s_1, \dots s_n$ be natural numbers with $\gcd(s_1, \dots, s_n) = 1$. Determine (or bound) the largest integer which can not be written as $k_1 s_1 + \dots + k_n s_n$ for some non-negative integers $k_1, k_2, \dots, k_n$.

A related question is, how many non-negative integers can not be written as $k_1 s_1 + \dots k_n s_n$? (When $n = 2$, this is $(s_1 - 1)(s_2 - 1)/2$).

Numerical Semigroups

A numerical semigroup S is a submonoid of $(\mathbb{N}, +)$ with $|\mathbb{N} \setminus S| < \infty$.

Or equivalently, $S = \langle s_1, \ldots, s_n \rangle$ for some $s_1, \ldots, s_n \in \mathbb{N}$ with $\gcd(s_1, \ldots, s_n) = 1$.

The set $G(S) := \mathbb{N} \setminus S$ is called the set of **gaps** of S.

The number $|G(S)|$ is called the **genus** of S (the Rees index of S in $\mathbb{N}$).

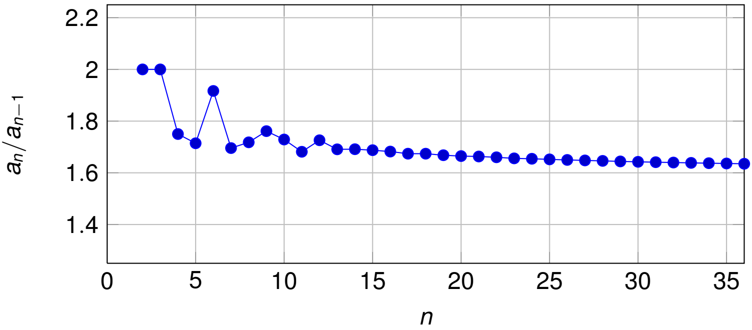
The element $f(S) := \max\{s \mid s \in G(S)\}$ is called the **Frobenius** of S.

The element $m(S) := \min\{s \mid s \in S\}$ is called the **multiplicity** of S.

A numerical semigroup is called **ordinary** if $f(S) < m(S)$, i.e. all the gaps are at the beginning.

How many numerical semigroups are there of a given genus?

n	1	2	3	4	5	6	7	8	9	10
a_n	1	2	4	7	12	23	39	67	118	204
a_n/a_{n-1}		2	2	1.75	1.71	1.91	1.69	1.71	1.76	1.72



Fibonacci like growth

In 2008 Maria Bras-Amorós conjectured that a_n has 'Fibonacci like' growth.

(Fibonacci-like behavior of the number of numerical semigroups of a given genus. Semigroup Forum 76, 2008).

In 2009 Bras-Amorós gave upper and lower bounds

(Bounds on the number of numerical semigroups of a given genus. J. Pure Appl. Algebra 213, 2009).

In 2013 Alex Zhai proved that $\lim_{n\to\infty} \frac{a_n}{\phi^n} = C$ where $\phi = \frac{1+\sqrt{5}}{2}$ and C is a constant.

(Fibonacci-like growth of numerical semigroups of a given genus. Semigroup Forum 86, 2013).

Free semigroups of higher rank

Given a finite alphabet $X_r := \{x_1, \ldots, x_r\}$, let $FS_r := X_r^+$ denote the free semigroup of rank r.

Let $a(n, r)$ denote the number of Rees index n subsemigroups of FS_r.

Then $a(n, 1)$ has 'Fibonacci like' growth.

Can we determine the rate of growth or find bounds for $a(n, r)$ with $r \geq 2$?

Lemma

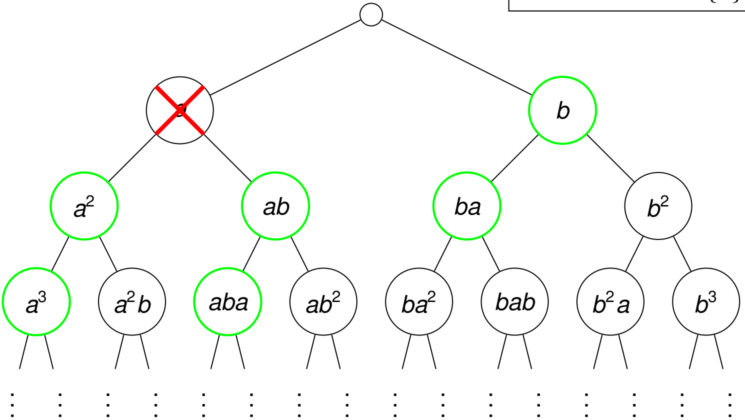
Every finite (Rees) index subsemigroup of FS_r has a finite unique minimal generating set.

Using the shortlex order, we can define Frobenius, multiplicity, ordinary etc.

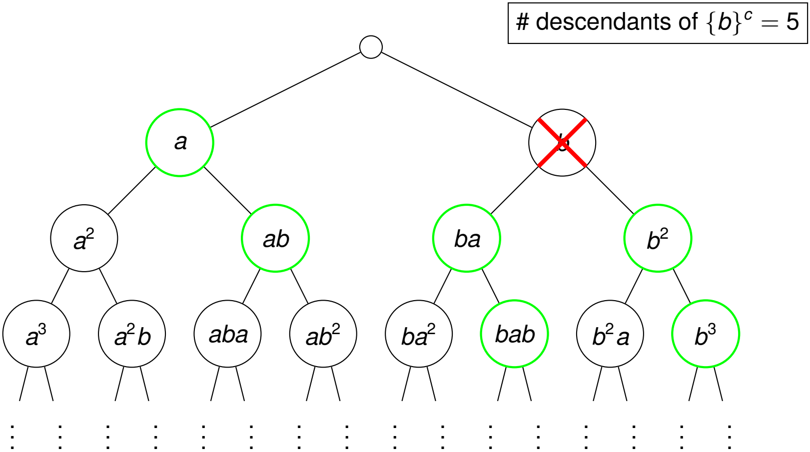
We can construct a tree of all finite index subsemigroups of FS_r by considering minimal generators larger than the Frobenius.

Example for $r = 2$

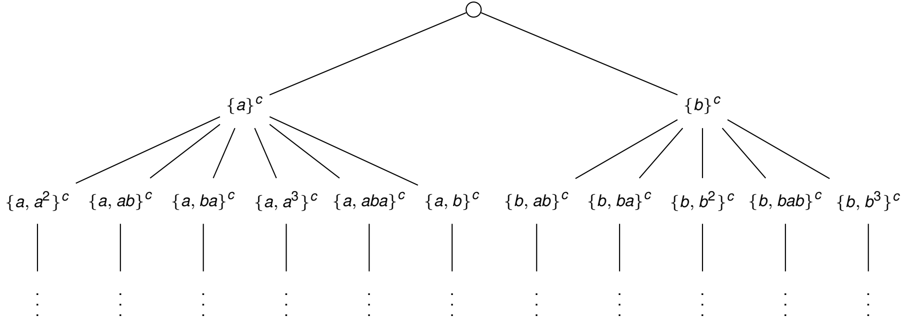
$\boxed{\#\text{ descendants of } \{a\}^c = 6}$



Example for $r = 2$

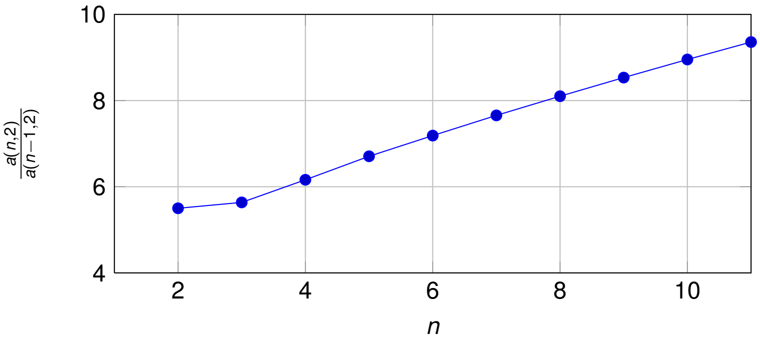


Tree of all finite index subsemigroups of FS_2



Number of index n subsemigroups of FS_2

n	1	2	3	4	5	6	7	8
$a(n,2)$	2	11	62	382	2562	18413	140968	1142004
$\frac{a(n,2)}{a(n-1,2}$		5.5	5.64	6.16	6.71	7.19	7.66	8.10



Ordinary subsemigroups

Levels k to $2k-1$: $\quad (r^k - i) + r^{k+1} + r^{k+2} \dots r^{2k-1} = r^k\left(\frac{r^k-1}{r-1}\right) - i$

Level $2k$: $\quad \sum_{j=0}^{i-1}(2(r^k - j) - 1) = 2ir^k - i^2$

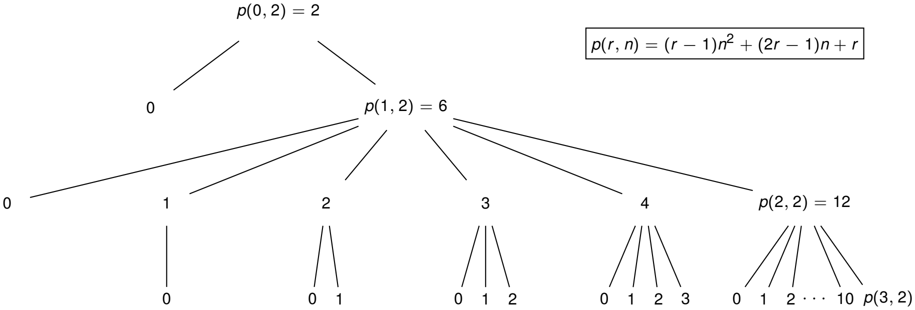
Level $2k+1$: $\quad ri^2$

Let $p(n, r)$ be the number of minimal generators (= number of descendants) of the index n ordinary subsemigroup of FS_r. Then:

$$
\begin{aligned}
p(n,r) &= r^k\left(\frac{r^k-1}{r-1}\right) - i + 2ir^k - i^2 + ri^2 \\
&= (r-1)\left(\frac{r^k-r}{r-1} + i\right)^2 + (2r-1)\left(\frac{r^k-r}{r-1} + i\right) + r \\
&= (r-1)n^2 + (2r-1)n + r
\end{aligned}
$$

Lower bound

$$\boxed{p(r,n) = (r-1)n^2 + (2r-1)n + r}$$



$$L(n,r) = \sum_{i=0}^{n} \binom{p(n-i,r)-1}{i} \geq \binom{p(n-(n/2),r)-1}{n/2} \geq \left(\frac{p(n/2,r)-1}{n/2}\right)^{n/2}$$

$$= \left(\frac{(r-1)\left(\frac{n}{2}\right)^2 + (2r-1)\frac{n}{2} + (r-1)}{n/2}\right)^{n/2} = \left(\sqrt{(r-1)\frac{n}{2} + (2r-1) + \frac{2r-2}{n}}\right)^{n}$$

This is super exponential for $r \geq 2$. (Note that $L(n,1) = F_n$).

Upper bound

We first show:

Lemma

Given an index n subsemigroup S of FS_r, the number of minimal generators of $(S \setminus \{m\}) \cup \{f\}$ is no less than the number of minimal generators of S.

This gives us:

Corollary

For a fixed index n, the ordinary subsemigroup of FS_r has the maximum number of descendants.

So for an upper bound, assume every index n subsemigroup of FS_r has $p(n, r)$ descendants.

Upper bound

$$\begin{aligned}
U(n,r) &= \prod_{k=0}^{n-1}\Big((r-1)k^2+(2r-1)k+r\Big)\\
&= \prod_{k=0}^{n-1}\Big((r-1)(k+1)^2+(k+1)\Big)\\
&= \prod_{k=1}^{n}\Big((r-1)k^2+k\Big) \le \prod_{k=1}^{n}\Big((r-1)k^2+(r-1)k\Big) \qquad \text{for } r \ge 2\\
&= (r-1)^n\prod_{k=1}^{n}(k(k+1))\\
&= (r-1)^n(n+1)(n!)^2\\
&\le (r-1)^n(n+1)\left(e\left(\frac{n+1}{e}\right)^{n+1}\right)^2 \quad = \quad \left(\frac{r-1}{e^2}\right)^n(n+1)^{2n+3}
\end{aligned}$$

and so

$$\left(\sqrt{(r-1)\frac{n}{2}+(2r-1)+\frac{2r-2}{n}}\right)^n \le a(n,r) \le \left(\frac{r-1}{e^2}\right)^n(n+1)^{2n+3}.$$

Zeta Functions

Given a sequence $a(n)$, with $s(n) = \sum_{i=1}^{n} a(n)$ the sequence of partial sums,

$$\text{let} \quad Z_a(s) := \sum_{n=1}^{\infty} a(n)n^{-s}.$$

This is called the "Dirichlet series associated to a(n)".

If $a(n)$ is polynomially bounded then $Z_a(s)$ converges for $Re(s) > \alpha$ where $\alpha = \inf\{d \in \mathbb{R} \mid \exists c \in \mathbb{R}_{>0} \text{ s.t. } s(n) < cn^d \text{ for all } n\}$.

The line $\mathrm{Re}(s) = \alpha$ is called the abscissa of convergence.

Examples:

1. If $a(n) = 1$ then $Z_a(s) := \zeta(s)$ the Riemann Zeta function which converges $Re(s) > 1$.
2. If $a(n) = \phi(n)$, then $Z_a(s) := \frac{\zeta(s-1)}{\zeta(s)}$, which converges $Re(s) > 2$.
3. If $a(n) = \phi(n)n^{-s}$, then $Z_a(s) := \frac{\zeta(2s-1)}{\zeta(2s)}$, which converges $Re(s) > 2/2 = 1$.

Zeta Functions

Let $t_{(n,2)}$ be the number of 2 generated index n subsemigroups of $\mathbb{N}$, and $s_{(n,2)} = \sum_{k=1}^{n} t_{(k,2)}$.

Given some $S = \langle a, b \rangle \subseteq \mathbb{N}$ with $\gcd(a, b) = 1$, then $b = ja + i$ for some $j \geq 1$ and $i \in U_a$ coprime to a, where $|U_a| = \phi(a)$.

Then $g(a, b) = |G(S)| = (a - 1)(b - 1)/2$ is the index or genus of S.

Upper bound for Zeta function of $t_{(n,2)}$

For the upper bound, we count correctly but *underestimate* index:
$g(a, aj + 1) \leq g(a, aj + i)$

$$\begin{aligned}
\sum_{n=1}^{\infty} t_{(n,2)} n^{-s} &= \sum_{a=2}^{\infty} \sum_{j=1}^{\infty} \sum_{i \in U_a}^{\infty} g(a, ja + i)^{-s} \\
&= \sum_{a=2}^{\infty} \sum_{j=i}^{\infty} \phi(a) \left((a-1)(aj + 1 - 1)/2\right)^{-s} \\
&= 2^s \zeta(s) \sum_{a=2}^{\infty} \phi(a) \left(a^2 - a\right)^{-s} \\
&\sim \sum_{a=2}^{\infty} \phi(a) a^{-2s} = \frac{\zeta(2s-1)}{\zeta(2s)}, \qquad \text{which converges for } Re(s) = 1.
\end{aligned}$$

Lower bound for Zeta function of $t_{(n,2)}$

For the lower bound we *under count* with correct index:

$g(\langle 2, 2n+1 \rangle) = n \rightsquigarrow \zeta(s)$, which also converges for $\mathrm{Re}(s) = 1$.

Therefore $s_{(n,2)}$ has 'linear growth', that is $\forall \epsilon > 0, \exists c$ s.t. $s_{(n,2)} < cn^{1+\epsilon}$.

Theorem

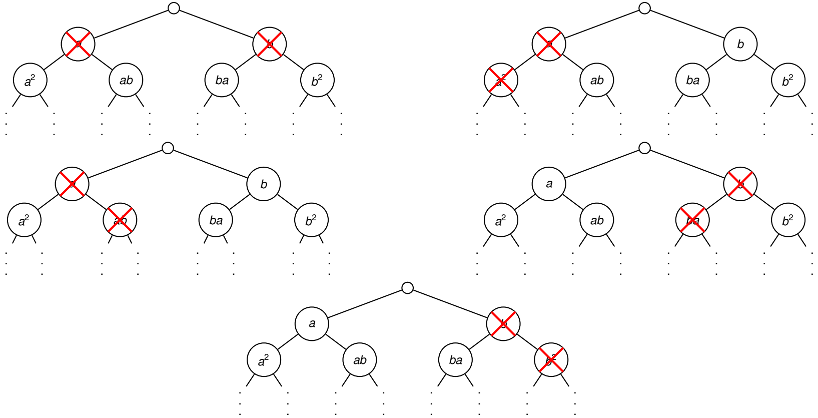
'degree $d-1$ growth' $\leq s_{(n,d)} \leq$ *'degree d growth'.*

Computational results

r \ n	1	2	3	4	5	6	7
1	1	2	4	7	12	23	39
2	2	11	62	382	2562	18413	140968
3	3	27	250	2568	28746	347691	4495983
4	4	50	644	9209	143416	2415078	43532832
5	5	80	1320	24150	480736	10340800	238120365
6	6	117	2354	52437	1269738	33192442	928558122
7	7	161	3822	100317	2859878	87935351	2892046165
8	8	212	5800	175238	5746592	203079088	7672012360
9	9	270	8364	285849	10596852	423019929	18042714315
10	10	335	11590	442000	18274722	813079415	38632533180
11	11	407	15554	654742	29866914	1465238951	76729376515
12	12	486	20332	936327	46708344	2504570454	143291607432
13	13	572	26000	1300208	70407688	4096363050	254187917217
14	14	665	32634	1761039	102872938	6453945820	431689558638
15	15	765	40310	2334675	146336958	9847206595	706238357145

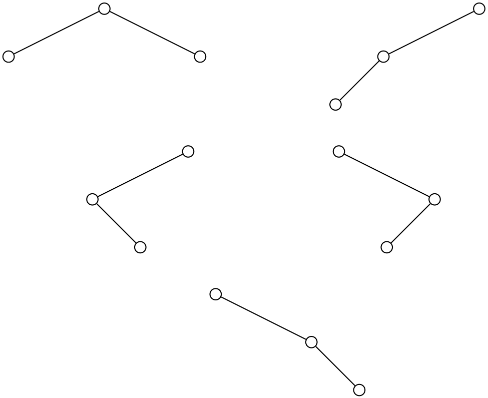
Ideal growth

Let Λ be a (Rees) index n (right) ideal of FS_r, then it is clear that no gap of Λ can have a non-gap as a parent. For example, there are 5 (right) ideals of FS_2 of Rees index 2.



Ideal Growth

Each of these correspond to a rooted binary tree with 3 vertices.



The number of these is precisely the 3^{rd} Catalan number.

Ideal growth

Let $a(n,r)$ be the number of Rees index n (right/left) ideals of FS_r. (These are the same number as the map $FS_r \to FS_r$, $w \mapsto rev(w)$ is an anti-isomorphism).

Then $a(n,r)$ is precisely the number of rooted r-ary trees with $n+1$ vertices.

$a(n+1,r) = \frac{1}{(r-1)n+1}\binom{r\ n}{n}$.

These are just the 'Fuss-Catalan' numbers.