Coherent monoids

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Introduction

- **Coherency** is a finitary property for monoids;
- it arises from **model theoretic** considerations of S-acts;
- it seems to be **awkward** to investigate in a consistent way.

Finitary properties

A **finitary property** is a property defined for monoids that is satisfied by all finite monoids.

Examples

1. **Chain** conditions on (one-sided) ideals: e.g. the d.c.c. on principal right ideals M_R - if
$$a_1 S \supseteq a_2 S \supseteq \dots$$
then the chain stabilises with
$$a_n S = a_{n+1} S = \dots;$$
2. **Finitely generated** - $S = \langle a_1, \dots, a_k \rangle$;
3. **Automatic** - S described using regular languages (finite state automata) for the set and multiplication;
4. conditions arising from **'Homological classification'** of monoids - guaranteeing that certain classes of S-acts coincide e.g. **perfection** for monoids;
5. conditions arising from **model theoretic** considerations of monoids and S-acts (such as **coherency**).

Outline

1. A brief introduction to S-acts.
2. What is coherency?
3. Which monoids are coherent?
4. Closure properties of the class of coherent monoids.
5. Further questions/unfinished theorems.

1. A brief introduction to S-acts: S-acts

Throughout, S is a **monoid**.

A **(right)** S-**act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Note any right ideal of S is an S-act - in particular, S is an S-act.

Beware: an S-act is also called an:
S-action, S-system, S-set, S-operand or (S)-polygon

The category **Act-S**

A and B are S-acts, a map $\alpha : A \to B$ is an S-**morphism** if

$$(as)\alpha = (a\alpha)s$$

for all $a \in A, s \in S$.

S-acts, together with S-morphisms, form a category **Act-S**

Act-S is a well-behaved category: epis are onto, monos are one-one, products are Cartesian products, coproducts are disjoint union

Note: If U, V are S-subacts of A, then so are $U \cap V$ and $U \cup V$.

Finitely generated S-acts

A is **monogenic** if $A = aS = \{as : s \in S\}$ for some $a \in A$.

An S-act A is **finitely generated** if

$$A = a_1 S \cup a_2 S \cup \ldots \cup a_n S$$

for some $a_1, \ldots, a_n \in A$.

A **congruence** on A is an equivalence that is compatible with the action of S.

If $H \subseteq A \times A$, then $\langle H \rangle$, the **congruence generated by H** is given by $a \langle H \rangle b$ iff $a = b$ or $\exists$ an H-sequence

$$a = c_1 t_1,\ d_1 t_1 = c_2 t_2, \ldots, d_n t_n = b,$$

where $(c_i, d_i) \in H \cup H^{-1}$, $t_i \in S$.

Monogenic S-acts are of the form S/ρ where ρ is a right congruence on S.

Finitely presented S-acts

The **free** S-act on a set X is

$$F_X^S = \bigcup_{x \in X} \{xs : x \in X, s \in S\}$$

where $xs = yt$ iff $x = y, s = t$. The action is given by

$$(xs)t = x(st),$$

for $x \in X, s, t \in S$. So

$$F_X^S = \bigcup_{x \in X} xS.$$

Note that S is the free monogenic S-act.

A is **finitely presented** if $A \cong F_X^S/\rho$, where X is a finite set and ρ is finitely generated (i.e. $\rho = \langle H \rangle$ for some finite $H \subseteq A \times A$).

2. What is coherency?

A monoid S is **right coherent** if every finitely generated S-subact of every finitely presented S-act is finitely presented.

Consider $A = F_X^S/\rho$ where X is finite and ρ is finitely generated.

A typical element of A looks like $[xu] = [x]u$.

Let $B = [x_1]u_1S \cup \ldots \cup [x_k]u_kS$ be finitely generated. For S to be right coherent, we need the kernel of

$$\begin{array}{rcl} y_1S \cup \ldots \cup y_kS & \to & B \\ y_i & \mapsto & [x_i]u_i \end{array}$$

to be finitely generated.

In particular, for any $s \in S$, considering $S \to sS, 1 \mapsto s$ we have

$$\mathbf{r}(s) = \{(u, v) \in S \times S : su = sv\}$$

is finitely generated.

Why are we interested in coherency?

Theorem **Wheeler 76** The following conditions are equivalent:

1. The theory of S-acts (in the appropriate first order language) has a **model companion**;
2. the class of existentially closed S-acts $\mathcal{E}$ is **axiomatisable**;
3. S is **right coherent**.

If any of the above hold, the sentences of the model companion axiomatise $\mathcal{E}$.

Question Which monoids are right coherent?

A result that helps us check when S is right coherent

Let A be an S-act and let $c \in A$. We define

$$\mathbf{r}(c) = \{(u, v) \in S \times S : cu = cv\}.$$

Notice that $\mathbf{r}(c)$ is a right congruence on S; it is the **right annihilator** congruence of c.

Theorem **G 19[redacted]** The f.a.e. for S:

1. S is right coherent;
2. every finitely generated S-subact of every S/ρ, where ρ is finitely generated, is finitely presented;
3. for every finitely generated right congruence ρ on S and every $a, b \in S$ we have $\mathbf{r}([a])$ is finitely generated, and $[a]S \cap [b]S$ is finitely generated.

What do we really mean by $r([a])$ being finitely generated?

Recall

$$\mathbf{r}(c) = \{(u, v) \in S \times S : cu = cv\}$$

so that here

$$\begin{aligned} \mathbf{r}([a]) &= \{(u, v) \in S \times S : [a]u = [a]v\} \\ &= \{(u, v) : S \times S : au \,\rho\, av\}. \end{aligned}$$

3. Which monoids are right coherent?

Observation Finite monoids are right coherent.

Theorem **G 19[redacted]** Any Clifford monoid S is right coherent.

Note that this means groups and semilattices are right coherent.

Proof.

Let $\rho = \langle H \rangle$ where H is finite, and let $a \in S$. Note that

$$a1 = a(a'a)$$

so $(1, a'a) \in r([a])$. Also, for any $(u, v) \in H$ we have

$$aa'u = uaa' \, \rho \, vaa' = aa'v$$

so that $(a'u, a'v) \in r([a])$. Hence

$$K = \{(1, a'a)\} \cup \{(a'u, a'v) : (u, v) \in H\} \subseteq r([a]).$$

On the other hand, if $(h, k) \in r([a])$, we have an H-sequence

$$ah = c_1 t_1, d_1 t_1 = c_2 t_2, \ldots, d_n t_n = ak.$$

Then $h = 1h \, \langle K \rangle \, a'ah = a'c_1 t_1 \, \langle K \rangle \, a'd_1 t_1 = a'c_2 t_2 \, \langle K \rangle \ldots \, \langle K \rangle \, a'd_n t_n = a'ak \, \langle K \rangle \, 1k = k$, so that $h \, \langle K \rangle \, k$. □

Free objects and coherency

Theorem **Choo, Lam, Luft 1972** Free rings are right coherent

Proposition **G, 1992** Free commutative monoids are right coherent.

Theorem **G, Hartmann, Ruskuc 2012/13** Free monoids are right coherent.

Free objects and coherency: G, Hartmann, Ruskuc 2012/13

Recall that any free group $FG(X)$ is right coherent.

Proposition Free inverse monoids are **not** right coherent.

Recall that if X is a set, then the free inverse monoid $FI(X)$ has set

$$\{(A, a) : A \subseteq FG(X), A \text{ finite and prefix closed}, a \in A\}$$

with multiplication given by

$$(A, a)(B, b) = (A \cup aB, ab).$$

Note that $(A, a)^{-1} = (a^{-1}A, a^{-1})$ and $(A, a)(A, a)^{-1} = (A, 1) = (A, a)^+$.

Theorem **Fountain, 1981** The free left ample monoid on X is the submonoid

$$FLA(X) = \{(A, a) \in FI(X) : A \subseteq X^*\}$$

of $FI(X)$.

Theorem Free left ample monoids are right (but not left) coherent.

Free objects and coherency: G, Hartmann, Ruskuc 2012/13

Idea of proof that X^* is right coherent:
Suppose that $\rho = \langle H \rangle$ where H is finite, and $au \, \rho \, av$.
Then there is a ρ-sequence

$$au = c_1 t_1, \; d_1 t_1 = c_2 t_2, \dots, d_n t_n = av.$$

By cancellation we can assume one of $u = t_0, t_1, \dots, t_n, t_{n+1} = v$ is ϵ.
If $t_k = \epsilon$, then $au \, \rho \, c_k$. Consider

$$au = c_1 t_1, \; d_1 t_1 = c_2 t_2, \dots, d_{k-2} t_{k-2} = c_{k-1} t_{k-1}, (d_{k-1} t_{k-1} = c_k)$$

By cancellation, there exists $z \in X^*$ with

$$u = u'z, t_i = t_i' z \text{ and } t_j' = \epsilon.$$

Then $au' \, \rho \, c_j$.
If $c_k = c_j$, then

$$au' \, \rho \, au \, \rho \, av$$

and use induction on length of words.

Right coherent and right noetherian monoids

A monoid S is **weakly right noetherian** if every right ideal is finitely generated.

S is **right noetherian** if every right congruence is finitely generated.

Theorem **Normak 77** If S is right noetherian, it is right coherent.

Example **Fountain 92** There exists a weakly right noetherian S which is not right coherent.

How is being weakly right noetherian related to being right coherent?

Proposition **G, 2005** Regular monoids for which every right ideal is principal are right coherent, e.g. **bicyclic monoids** and **Bruck-Reilly** semigroups $BR(G, \theta)$, where G is a group.

Right coherent and right noetherian monoids: G, Hartmann, Ruskuc, Yang 2012/13

Theorem (a) Regular monoids for which certain 'annihilator right ideals' are finitely generated are right coherent.

e.g. $(\mathbb{Z} \times \mathbb{Z})^1$ with 'bicyclic' multiplication.

Consequently, weakly right noetherian regular monoids are right coherent.

(b) Brandt semigroups with 1 adjoined are right coherent and such that the 'annihilator ideals' need not be finitely generated.

4. Closure properties of the class of coherent monoids: G,H,R,Y 2012/13 Negative results

Let $\mathcal{RC}$ be the class of right coherent monoids.

Result 1 The class $\mathcal{RC}$ is not closed under direct products: the direct product of two free monoids is not right coherent.

Result 2 The class $\mathcal{RC}$ is not closed under submonoids.

Proof We have that $X^* \times X^*$ is a submonoid of $FG(X) \times FG(X)$ and the latter is a group, so right coherent.

Result 3 The class $\mathcal{RC}$ is not closed under (a) morphisms (b) inverse images.

Proof (a) X^* is right coherent and not all monoids are;
(b) some inverse monoids are right coherent by $FI(X)$ is not.

Closure properties of the class of coherent monoids: G,H,R,Y 2012/13 Positive results

Recall that a **retract** of a monoid S is a submonoid T such that $T = \mathrm{Im}\,\theta$ for some idempotent morphism $\theta : S \to S$.

Result 4 The class $\mathcal{RC}$ is closed under:

(a) retracts;

(b) monoid $\mathcal{J}$-classes.

Closure properties of the class of coherent monoids: G,H,R,Y 2012/13
Positive results

Let M be a monoid: **Two common constructions**

Brandt extension Let I be a non-empty set. Then

$$\mathcal{B}^0(M, I) = (I \times M \times I) \cup \{0\}$$

is a semigroup where all products are zero except

$$(i, m, j)(j, n, h) = (i, mn, h).$$

Bruck-Reilly extension
Let $\theta : M \to M$ be a morphism, then

$$BR(M, \theta) = \mathbb{N}^0 \times M \times \mathbb{N}^0$$

with multiplication

$$(a, m, b)(c, n, d) = (a - b + t, m\theta^{t-b} n\theta^{t-c}, d - c + t)$$

where $t = \max\{b, c\}$.

Closure properties of the class of coherent monoids: G,H,R,Y 2012/13 Positive results

Result 5 Let M be a monoid. If (a) $(\mathcal{B}^0(M,I))^1$ or (b) $BR(M,\theta)$ is right coherent, then so is M.

Result 6 If M is right coherent, then so is $(\mathcal{B}^0(M,I))^1$.

5. Further questions/unfinished theorems

1. If M is right coherent, is $BR(M, \theta)$?
2. Closure of $\mathcal{RC}$ under Rees quotients, Rees matrix construction, up-down results of adding and subtraction zeroes and identities, 0-direct unions, certain direct and semidirect products.
3. Right coherency for semigroups.
4. A consistent (coherent) approach - working within $\mathcal{D}$-classes.
5. Connection with direct products/ultraproducts of flat **left** S-acts.
6. $\mathcal{T}_X, \mathcal{I}_X$?