Free idempotent-generated semigroups

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Idempotent-generated semigroups

S - semigroup, $E = E(S)$ - idempotents of S.

Definition. S is *idempotent-generated* if $S = \langle E \rangle$.

Example 1 (Howie 1966)

$\mathcal{T}_n$ - full transformation monoid, $S(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n : \operatorname{rank} \alpha < n\}$.

Example 2 (J.A. Erdös, 1967, Laffey, 1973)

$M_n(D)$ - full linear monoid, $S(M_n(D)) = \{A \in M_n(D) : \operatorname{rank} A < n\}$.

Example 3 (Fountain and Lewin, 1992)

A - independence algebra, $S(\operatorname{End} A) = \{\alpha \in \operatorname{End}(A) : \operatorname{rank} \alpha < n\}$.

Free idempotent-generated semigroups

S - semigroup, $E = E(S)$ - idempotents of S.

E - satisfies a number of axioms: that of a *biordered set.*

S - regular, an extra axiom holds: that of a *regular biordered set.*

Nambooripad and Easdown (regular) biordered sets $\longleftrightarrow$ (regular) semigroups.

$\mathcal{A}$: family of all semigroups whose biordered set are isomorphic to a fixed E.

Within this family, there exists an unique free object $\mathrm{IG}(E) \in \mathcal{A}$, *free idempotent-generated semigroup* defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Free idempotent-generated semigroups

Facts

(I) $\mathrm{IG}(E) = \langle \overline{E} \rangle$.

(II) The natural map $\phi : \mathrm{IG}(E) \to S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E(S) \rangle$.

(III) The restriction of ϕ to the set of idempotents of $\mathrm{IG}(E)$ is a bijection.

(IV) The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in $\mathrm{IG}(E)$ and the corresponding set in $S' = \langle E(S) \rangle$.

Maximal subgroups of IG(E)

Question: Which groups can arise as the maximal subgroups of free idempotent-generated semigroups IG(E)?

Note: For any semigroup S, the maximal subgroup with identity e of S is the $\mathcal{H}$-class H_e of S.

Idea: Compare the $\mathcal{H}$-class $H_{\overline{e}}$ in IG(E) and H_e in $S = \langle E \rangle$.

Work of Pastijn (1977, 1980), Nambooripad and Pastijn (1980), McElwee (2002) led to a conjecture that all these groups must be free groups. .

Brittenham, Margolis and Meakin (2009)

$\mathbb{Z} \oplus \mathbb{Z}$ can be a maximal subgroup of IG(E), for some E.

Gray, Ruskuc (2011)

Every group occurs as the maximal subgroup of IG(E) by using a general presentation and a special choice of E.

Full transformation monoids and full linear monoids

$\mathcal{T}_n$ - full transformation monoid, $E = E(\mathcal{T}_n)$ - its biordered set.

Gray and Ruskuc (2011)

$\operatorname{rank} e = r < n-1$, $H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.

Brittenham, Margolis and Meakin (2010)

$M_n(D)$ - full linear monoid, $E = E(M_n(D))$ - its biordered set.

$\operatorname{rank} e = 1$ and $n \geq 3$, $H_{\overline{e}} \cong H_e \cong D^*$.

Dolinka and Gray (2011)

$\operatorname{rank} e = r < n/3$ and $n \geq 4$, $H_{\overline{e}} \cong H_e \cong GL_r(D)$.

Note $\operatorname{rank} e = n-1$, $H_{\overline{e}}$ is free; $\operatorname{rank} e = n$, $H_{\overline{e}}$ is trivial.

Endomorphism monoids of free G-acts

Observation Sets and vector spaces are examples of *independence algebras*, as is any rank n free (left) G-act $F_n(G)$.

Let G be a group and $n \in \mathbb{N}$. The *free G-act $F_n(G)$* is given by

$$F_n(G) = Gx_1 \cup Gx_2 \cup \cdots \cup Gx_n$$

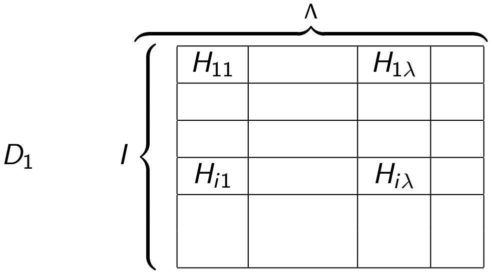
where $g(hx_i) = (gh)x_i$.

Note $S(\operatorname{End} F_n(G)) = \{\alpha \in \operatorname{End} F_n(G) : \operatorname{rank} \alpha < n\} = \langle E \rangle$.

$$\begin{aligned} \alpha \,\mathcal{R}\, \beta &\Leftrightarrow \ker \alpha = \ker \beta \\ \alpha \,\mathcal{L}\, \beta &\Leftrightarrow \operatorname{im} \alpha = \operatorname{im} \beta \\ \alpha \,\mathcal{D}\, \beta &\Leftrightarrow \operatorname{rank} \alpha = \operatorname{rank} \beta. \end{aligned}$$

Endomorphism monoids of free G-acts: rank $e = 1$

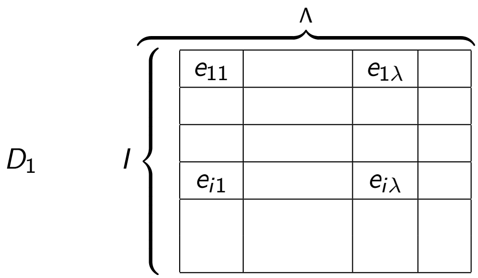
rank $e = 1$, D_1 - $\mathcal{D}$-class of e in End $F_n(G)$, $1 \in I \cap \Lambda$.



Note D_1 is a completely simple semigroup and $H_{11} \cong H_{i\lambda} \cong G$.

Endomorphism monoids of free G-acts: rank $e = 1$

$e_{i\lambda}$ - the identity of $H_{i\lambda}$, $H_{11} = H_{e_{11}}$.

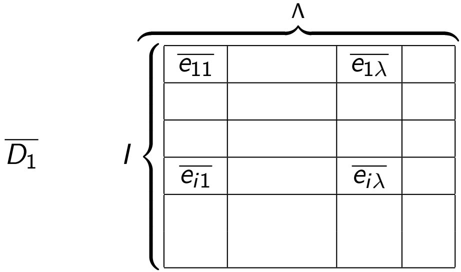


Hall Let $E_1 = \{e_{i\lambda} : (i, \lambda) \in I \times \Lambda\}$. Then $D_1 = \langle E_1 \rangle$.

$H_{11} = \{e_{11} e_{i\lambda} e_{11} : i \in I, \lambda \in \Lambda\}$.

$\overline{D_1}$- $\mathcal{D}$-class of $\overline{e}$ in $\mathrm{IG}(E)$.

By Facts (I-IV), $\overline{D_1}$ is completely simple.



$$H_{\overline{e_{11}}} = \langle \overline{e_{11}}\, \overline{e_{i\lambda}}\, \overline{e_{11}} : (i, \lambda) \in I \times \Lambda \rangle_{Gp} = \overline{H_{11}}.$$

Endomorphism monoids of free G-acts: rank $e = 1$

We show that

1. $e_{11}e_{i\lambda}e_{11} = e_{11}$ implies that $\overline{e_{11}}\,\overline{e_{i\lambda}}\,\overline{e_{11}} = \overline{e_{11}}$.
2. $e_{11}e_{i\lambda}e_{11} = e_{11}e_{j\mu}e_{11}$ implies that $\overline{e_{11}}\,\overline{e_{i\lambda}}\,\overline{e_{11}} = \overline{e_{11}}\,\overline{e_{j\mu}}\,\overline{e_{11}}$.
3. $$(e_{11}e_{i\lambda}e_{11})(e_{11}e_{k\nu}e_{11}) = e_{11}e_{j\mu}e_{11}$$
   implies that
   $$(\overline{e_{11}}\,\overline{e_{i\lambda}}\,\overline{e_{11}})(\overline{e_{11}}\,\overline{e_{k\nu}}\,\overline{e_{11}}) = \overline{e_{11}}\,\overline{e_{j\mu}}\,\overline{e_{11}}.$$
4. Similarly for inverses.
5. It follows easily that $\overline{H_{11}} \cong H_{11} \cong G$.

Corollary Every group occurs as $H_{\overline{e}}$ in some $\mathrm{IG}(E)$

Note Our proof is very easy; we do not need presentations; we use a natural biordered set.

rank $e = 2$ and $n \geq 4$, $H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_2(G)$

rank $e = r \leq n/3$ and $n \geq 3$, $H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_r(G)$

Note rank $e = 2$ and $n = 3$, H is free.

Conjecture: rank $e = r < n - 1$, $H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_r(G)$ - we will need some new techniques.

rank $e = r$, D_r - $\mathcal{D}$-class of e, $H = H_e$

$D_r^0 \cong \mathcal{M}^0(H; I, \Lambda; P)$, $P = (p_{\lambda i})$.

$H_{i\lambda} = R_i \cap L_\lambda$, $K = \{(i, \lambda) \in I \times \Lambda : \ H_{i\lambda} \text{ is a group.}\}$.

$H_{\overline{e}}$ - $\mathcal{H}$-class of $\overline{e}$ in $\mathrm{IG}(E)$. $H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_r(G)$?

R. Gray and N. Ruskuc (2011)

$H_{\overline{e}}$ is defined by the presentation $\mathcal{P}$ with generators:

$$F = \{f_{i,\lambda} : \quad (i, \lambda) \in K\}.$$

Defining relations **R1**, **R2** and **R3**.

Idea: To find the relationship between $f_{i,\lambda}$ and $p_{\lambda i}$.

Proofs

(I) $p_{\lambda i} = e$ implies $f_{i,\lambda} = 1$.

(II) $p_{\lambda i} = p_{\mu j} = A \in \operatorname{Aut} F_r$ implies $f_{i,\lambda} = f_{j,\mu}$.

Proof: (1) If $p_{\lambda i} = p_{\mu j} = A$ with one of the following simple forms, then $f_{i,\lambda} = f_{j,\mu}$.

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & ax_k & x_{k+1} & \cdots & x_r \end{pmatrix}$$

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ ax_1 & x_2 & \cdots & x_r \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_k & x_{k+1} & x_{k+2} & \cdots & x_r \\ x_2 & x_3 & \cdots & x_{k+1} & ax_1 & x_{k+2} & \cdots & x_r \end{pmatrix},$$

where $k \geq 2, m \geq 1$.

Proofs

(2) For every $A \in \mathrm{Aut}\, F_r$, we have

$$A = A_1 \cdots A_r,$$

where A_i has one of the simple forms, for all $i \in [1, r]$. Moreover, $p_{\lambda i} = A$ implies

$$f_{i,\lambda} = f_{i_1,\lambda_1} \cdots f_{i_r,\lambda_r},$$

where $p_{\lambda_1 i_1} = A_r, \cdots, p_{\lambda_r i_r} = A_1$.

(III) Denote generator $f_{i,\lambda}$ with $p_{\lambda i} = A$ by f_A.

(IV) For $r \leq n/3$, $f_{AB} = f_B f_A$, $f_{A^{-1}} = f_A^{-1}$.

(V) A mapping $\phi : H_{\overline{e}} \longrightarrow H_e$ defined by $f_A \mapsto A^{-1}$ is an isomorphism, i.e. $H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_r(G)$.

Future work $\cdots$

What if $n/3 < r < n-1$?

What will happen for independence algebras?

Question Let A be an independence algebra with rank $A = n$ and $S = S(\mathrm{End}(A)) = \{\alpha \in \mathrm{End}\, A : \mathrm{rank}\, \alpha < n\}$.

If $e \in E(S)$ with $\mathrm{rank}\, e = n-1$, then $H_{\overline{e}}$ is free.

Is it true that for $e \in E(S)$ with $\mathrm{rank}\, e \leq n-2$,

$$H_{\overline{e}} \cong H_e \cong \mathrm{Aut}\, F_r?$$

What if $\mathrm{rank}\, e = 1$?

Thank you !