Walkin' in free Inverse Monoids

(how to bring coal to newcastle ?)

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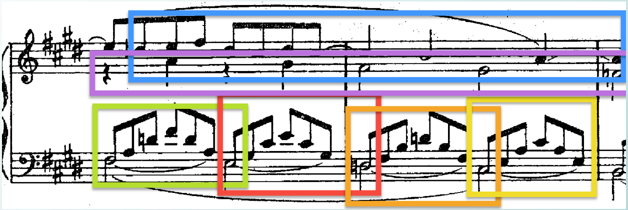
Music modeling

The structure of music is complex with mixed sequential, parallel and hierarchical features.



A theory of overlapping structures is needed for computer music analysis and/or production.

Observation

Inverse semigroup theory provides almost everything we need for music analysis [Jan12c] or for music design and production [BJM12].

1. Playground

Within free inverse monoids

Bi-rooted trees

The free inverse monoid and its Rees' quotients

$$\begin{tikzcd} \text{Walks} & (A+\bar{A})^* \arrow[r, two heads, "\eta"] \arrow[d, two heads, "\theta"'] & (A+\bar{A})^*/\theta^{-1}(\perp) \arrow[d, two heads, "\theta"'] \\ \text{BTrees} & FIM(A) \arrow[r, two heads, "\eta"] & FIM(A)/\perp \end{tikzcd}$$

Examples

Typical models defined by choosing adequate ideals.

- *directed trees* generated by $\perp = \langle \{a\bar{b}\}_{a,b\in A, a\neq b}\rangle$.
- McAlister *tiles* generated by $\perp = \langle \{a\bar{b}, \bar{a}b\}_{a,b\in A, a\neq b}\rangle$.

Birooted F-terms

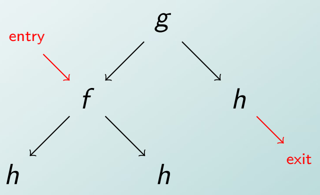
Signature

A finite alphabet of function names F and an arity mapping $\rho : F \to \mathcal{P}(E)$ with finite set E of argument names.

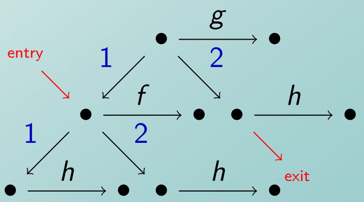
Example

$F = \{f, g, h\}$ with $\rho(f) = \{1, 2\}$, $\rho(g) = \{1, 2\}$ and $\rho(h) = \emptyset$.

F-tree $g(f(h, h), h)$



encoded as a bi-rooted tree



Birooted F-trees

Signature

A finite alphabet of function names F and an arity mapping $\rho : F \to \mathcal{P}(E)$.

Observation

Birooted F-trees can be embedded into $FIM(A)/\perp$ with alphabet $A = F + \{(f, e, g) \in F \times E \times F : e \in \rho(f)\}$ and $\perp$ the ideal of bad encodings, i.e. birooted trees that do not define a partial F-tree.

Observation

Complete (finite) birooted F-trees are minimal non zero elements in the natural order.

Examples

- $F = \{1\}$ with $\rho(1) = A$, essentially directed trees,
- $F = A$ with $\rho(a) = \{1\}$, essentially McAslister tiles.

2. Languages of birooted trees

Towards a birooted tree language theory

Three classical classes of languages

REC

Languages $L \subseteq FIM(A)$ recognizable by morphism, i.e. there is morphism $\varphi : FIM(A) \to S$ with finite S such that $L = \varphi^{-1}(\varphi(L))$.

RAT

Languages $L \subseteq FIM(A)$ definable by a rational expression, i.e. a finite combination of finite languages with sum $+$, product $\cdot$ and iterated product (Kleene star) $*$.

MSO

Languages $L \subseteq FIM(A)$ definable by an formula of Monadic Second Order logic (MSO), i.e. $L = \{x \in FIM(A) : x \models \varphi_L\}$ for some MSO definable characteristic property φ_L of L.

Separation result

Theorem (Buchi, Elgot)

Within A^* we have $REC = RAT = MSO$.

Theorem ([Jan13, DJ12])

Within $FIM(A)$ (or even M_A) we have $REC \subset RAT \subset MSO$ with strict inclusion.

3. Tile languages

McAlister monoid

Positive tiles encoded as triples $(u, v, w) \in A^* \times A^* \times A^*$

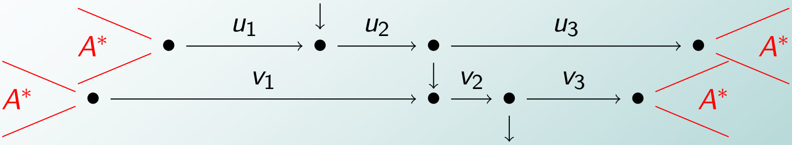
$$\bullet \xrightarrow{\quad u \quad} \overset{\downarrow}{\bullet} \xrightarrow{\quad v \quad} \underset{\downarrow}{\bullet} \xrightarrow{\qquad\qquad w \qquad\qquad} \bullet$$

and negative tiles encoded as $(uv, \bar{v}, vw) \in A^* \times \bar{A}^* \times A^*$

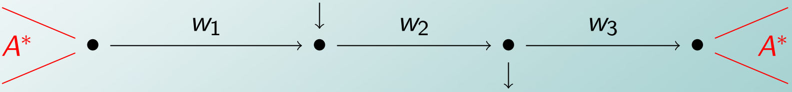
$$\bullet \xrightarrow{\quad u \quad} \underset{\downarrow}{\bullet} \xrightarrow{\quad v \quad} \overset{\downarrow}{\bullet} \xrightarrow{\qquad\qquad w \qquad\qquad} \bullet$$

Tiles product

Given two tiles encoded as $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$,



there is at most one tile $w = (w_1, w_2, w_3)$



- left match : $A^* w_1 = A^* u_1 \cap A^* v_1 \bar{u}_2$,
- right match : $w_3 A^* = v_3 A^* \cap \bar{v}_2 u_3 A^*$.

In that case we take $u \cdot v = w$ and otherwise we take $u \cdot v = 0$.
The resulting tile monoid, McAlister monoid, is denoted by M_A.

MSO

Operators on languages

- Sum: $X + Y = X \cup Y$,
- Product: $X \cdot Y = \{xy : x \in X, y \in Y\}$,
- Iterated product (star): $X^* = \sum_{k \in \mathbb{N}} X^k$,
- Idempotent proj.: $X^E = \{x \in X : xx = x\} = X \cap E(FIM(A))$.
- Inverse: $X^{-1} = \{x^{-1} : x \in X\}$.

Theorem (Robustness [Jan13], [Jan12d])

The class MSO of languages of tiles is closed under complement, sum, product, iterated product (star), inverses, idempotent projections.

MSO

Theorem (Simplicity [Jan13])

For every MSO language L, given L^+ (resp. L^-) the set of positive (resp. negative) tiles in L, we have:

$$L^+ = \sum_{k \in I} L_k \times C_k \times R_k \text{ and, resp. } L^- = \sum_{k \in J} (L_k \times C_k \times R_k)^{-1}$$

for finite I and J and regular word languages L_k, C_k and $R_k \subseteq A^*$.

Proof.

An MSO definable language of positive tiles is just an MSO definable language of words in $A_p^* A^* A_s^*$ with A_p and A_s two disjoint copies of A. □

MSO

Definition

Let E-RAT be the class of languages definable by means of sum, product, star and idempotent projection.

Theorem (MSO = E-RAT [DJ12])

Language $L \subseteq M_A$ is MSO if and only if it is definable by sum, product, star and idempotent projection of finite languages.

Proof.

E-RAT is closed under inverse operator and, for every regular L, C and $R \subseteq A^*$, $L \times C \times R = (1 \times L \times 1)^L \cdot (1 \times C \times 1) \cdot (1 \times R \times 1)^R$ with $X^L = \{x^{-1}x : x \in X\}$, $X^R = \{xx^{-1} : x \in X\}$ and that fact that $X^L = (X^{-1}X)^E = (X^{-1})^R$. $\square$

RAT

Fact

There is an ideal $\perp \subseteq (A+\bar{A})^*$ such that:

$$\begin{tikzcd} (A+\bar{A})^* \arrow[r, two heads, "/\perp"] \arrow[d, two heads, "\theta"] & (A+\bar{A})^*/\perp \arrow[d, two heads, "\theta"] \\ FIM(A) \arrow[r, two heads, "/\perp"] & M_A \end{tikzcd}$$

(walks)

(structures)

Theorem ([DJ12])

Language $L \subseteq M_A$ is RAT if and only if $L = \theta(W)$ for some regular language $W \subseteq (A+\bar{A})^*$.

RAT

Corollary

Language $L \subseteq M_A$ is RAT if and only if L recognizable by a finite walking automaton.

Proof.

Take the one way automaton on alphabet $A + \bar{A}$ that recognizes $W \subseteq (A + \bar{A})^*$ with $L = \theta(W)$.
Interpret it as a two-way automaton on tiles that recognizes $\theta(W) \subseteq M_A$. □

Corollary

The inclusion RAT $\subset$ MSO is strict as witnessed by $L = E(M_A)$ and a simple pumping argument (on the underlying walking automaton).

RAT

Corollary

If language $L \subseteq M_A$ is RAT then $L = \psi^{-1}(\psi(L))$ for some finite monoid S and relational morphism $\psi : M_A \to S$.

$$
\begin{tikzcd}
 & (A+\bar{A})^* \arrow[dl, "\theta"'] \arrow[dr, "\varphi"] & \\
M_A \arrow[rr, "\psi = \varphi \circ \theta^{-1}"] & & S
\end{tikzcd}
$$

Question

Does this lead to an interesting characterization of RAT ?

REC

Lemma

The inclusion REC $\subset$ RAT is strict as witnessed by $L = 1 \times ba^* \times 1$ that has a syntactic congruence of infinite index.

Theorem ([Jan13])

For every morphism $\varphi : M_A \to S$, every $s \in S - 0$, there are x and $y \in A^*$ such that: $\varphi^{-1}(s)$ is essentially a co-finite subset of tiles of the form (u, v, w) with ${}^{\omega}(xy) \geq_s u$, $v \in x(yx)^*$, $w \leq_p (yx)^{\omega}$.

Proof.

Let $\varphi : M_A \to S$ for some monoid S (even infinite). Let $s \in S - 0$. Then $\varphi^{-1}(s)$ is totally ordered both by left and right Green's preorder.

...and some combinatorics to conclude... $\square$

4. Walking in FIM(A)

Walk languages vs tree languages

Observation

Reading words of $(A+\bar{A})^*$ amount to walking on some underlying birooted trees.

Walk languages

Language $W \subseteq (A+\bar{A})^*$ is a walk language of the tree language $L \subseteq FIM(A)/\bot$ when $L + 0 = \eta \circ \theta(W) + 0$.

$$\begin{tikzcd}
(A+\bar{A})^* \arrow[rr, two heads, "\eta \circ \theta"] \arrow[dr, two heads, "\theta"'] & & FIM(A)/\bot \\
& FIM(A) \arrow[ur, two heads, "\eta"'] &
\end{tikzcd}$$

Question

How classes of tree languages in $FIM(A)/\bot$ are related with classes of the underlying walk languages in $(A+\bar{A})^*$?

Walking automata

Theorem ([Jan12d, DJ12])

- *REC = Strongly deterministic finite state walking Automata,*
- *RAT = Finite state walking automata,*
- *MSO = Many-Pebble finite state walking automata.*

Fact

$REC \neq RAT$ witnessed by ba^*.
$RAT \neq MSO$ witness by $E(FIM(A))$.

MSO and the pebble hierarchy

Idempotent projection

For every language X, let $X^E = \{x \in X : xx = x\}$.

k-rational languages

Language L is k-rational when either L is rational or $k > 0$ and L is a finite rational combination of languages of the form X or X^E with $X \in RAT^{k-1}$.

Fact

RAT^k is closed under inverses for every $k \in \mathbb{N}$.

Theorem ([Jan12d])

$REC \subset RAT = RAT^0 \subset RAT^1 \subseteq RAT^2 \subseteq \cdots \bigcup_k RAT^k \subseteq MSO$

probably with strict inclusions.

Theorem ([Jan13, DJ12])

Over tiles $REC \subset RAT \subset RAT^1 = MSO$.

5. Quasi-recognizability

A newcomer question

Fact

Within FIM(A), the class REC collapses.

Question

How to relax the notion REC into some (notion of) quasi-REC (QREC) in such a way MSO = QREC (in relevant case) ?

Ideas

1. relax morphism condition $\varphi(xy) = \varphi(x)\varphi(y)$ into premorphism condition $\varphi(xy) \leq \varphi(x)\varphi(y)$.
2. restrict to an adequate class of finite (ordered) monoid and premorphism in such a way that pre-images remain MSO definable.

Adhoc candidates for QREC

QREC points

Stable ordered monoid $(S, \leq)$ such that:

- $U(S) = \{x \leq 1\} \subseteq E(S)$, i.e. subunits are idempotents,
- for all $x \in S$, both $x_R = \bigwedge\{e \in U(S) : ex = x\}$ and $x_L = \{f \in U(S) : xf = x\}$ exist in $U(S)$,
- for all x and $y \in S$, if $x = x_R y x_L$ then $x \leq y$. .

QREC arrows

Premorphism $\varphi : FIM(A)/\bot \to (S, \leq)$, i.e. $\varphi(1) = 1$ and, for every x and y, if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ and $\varphi(xy) \leq \varphi(x)\varphi(y)$, such that:

- for every disjoint product $x \cdot y$, we have $\varphi(x \cdot y) = \varphi(x)\varphi(y)$,
- for every x, we have $\varphi(x_L) = (\varphi(x))_L$ and $\varphi(x_R) = (\varphi(x))_R$.

with $x_L = x^{-1}x$ and $x_R = xx^{-1}$ in $FIM(A)/\bot$.

QREC vs MSO

Let $M_A^+ = 0 + A^* \times A^* \times A^*$ be the submonoid of M_A of *positive tiles*.

Theorem ([Jan12b])

If $L \subseteq M_A^+$ is QREC then L is MSO.

Theorem ([Jan12b])

If $L \subseteq M_A^+$ is MSO and if tiles of L are plugged, i.e. with tiles of the form $(\#u, v, w\#)$ for some marker $\#$, then L is QREC.

$\mathcal{Q}$-expansion

Monoid $\mathcal{Q}$-expansion

Let S be a monoid. Let $\mathcal{Q}(S) = 0 + \mathcal{L}_S \times S \times \mathcal{R}_S$ with

$$(L, s, R) \cdot (M, t, N) = (L \cap (M)s^{-1}, st, t^{-1}(R) \cap N)$$

when compatible, and 0 otherwise.

Theorem ([Jan12a])

For every monoid S, monoid $\mathcal{Q}(S)$ ordered by $(L, s, R) \leq (M, t, N)$ when $L \subseteq M$, $s = t$ and $R \subseteq N$ is a stable U-semiadequate monoid.

Theorem

There is an embedding $\iota_A : M_A^+ \to \mathcal{Q}(A^*)$.

$\mathcal{Q}$-expansion

Morphism $\mathcal{Q}$-expansion

Let $\varphi : S \to T$. Let $\mathcal{Q}(\varphi) : \mathcal{Q}(S) \to \mathcal{Q}(T)$ defined, on every non zero positive tile (L, s, R), by

$$\mathcal{Q}(\varphi)(L, s, R) = (S\varphi(L), \varphi(s), \varphi(R)S)$$

and let $\eta_S : \mathcal{Q}(S) \to S^0$ defined by $\eta_S((L, s, R)) = s$.

Theorem ([Jan12a])

For every morphism $\varphi : S \to T$, mapping $\mathcal{Q}(\varphi)$ is a well-behaved premorphism (i.e. QREC arrows) and the following diagram commutes.

$$\begin{tikzcd}
\mathcal{Q}(S) \arrow[r, "\mathcal{Q}(\varphi)"] \arrow[d, "\eta_S"'] & \mathcal{Q}(T) \arrow[d, "\eta_T"] \\
S^0 \arrow[r, "\varphi"] & T^0
\end{tikzcd}$$

Theorem

For every plugged language $L \subseteq M^{+}_{A+\#}$, if L is MSO then, given $\varphi : A^* \to S$ "recognizing" $L \subseteq \#A^* \times A^* \times A^*\#$, then L is QREC by $\mathcal{Q}(\varphi) : M_A \to \mathcal{Q}(S)$.

6. Conclusion

Work in progress

Extending/developing QREC towards:

- languages of positive and negative tiles,
- languages of finite directed trees,
- languages of finite and infinite trees (completing $FIM(A)$ with infinite many rooted trees).

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