Products in Transformation Semigroups: work with the late John M. Howie

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From 1998, John Howie, Nik Ruskuc and I wrote a series of papers stemming from Howie's early efforts [6] on the subsemigroup $\langle E \rangle$ of T_X generated by the set of idempotents E. This blossomed in two directions:

1. the description of products of special sets in terms of parameters of the mappings involved and

2. 'rank' theorems where smallest sets of additional generators were identified to produce all of some type of transformation semigroup.

The first parameters, sets and relations that arose were:

The *shift* of $\alpha, S(\alpha) = \{x \in X : x\alpha \neq x\}$; the *fix* of α,
$F(\alpha) = X \setminus S(\alpha)$;
the *defect* of α, $D(\alpha) = X \setminus X\alpha$,
$\ker\alpha = \{(x,y) \in X \times X : x\alpha = y\alpha\}$.

The cardinals of these sets are denoted $s(\alpha)$, $f(\alpha)$, and $d(\alpha)$ while Ker α is the partition of X corresponding to ker α.

Another cardinal featured, the *collapse* of α: $c(\alpha) = |X \setminus T_\alpha|$, where T_α is any transversal of Ker α.

In his seminal 1966 paper John proved that $\langle E\rangle = F^1 \cup Q$ where
$F = \{\alpha \in T_X : d(\alpha) > 0 \text{ and } s(\alpha) < \aleph_0\}$,
$Q = \{\alpha \in T_X : s(\alpha) = d(\alpha) = c(\alpha) \geq \aleph_0\}$
Q is the set of *balanced* elements. In the paper [3] however arose the set of *semi-balanced elements* $B = \{\alpha \in T_X : d(\alpha) = c(\alpha)\}$ as did the sets $C = \{\alpha \in T_X : d(\alpha) \leq c(\alpha)\}$ and $D = \{\alpha \in T_X : d(\alpha) \geq c(\alpha)\}$. The main results of the paper are summarised in the following table.

	T	S	I	J	C	D	G	B	E	EE	
T	T	T	T	T	T	T	T	T	T	T	
S	T	S	J	J	C	P	S	C	C	C	
I	T	T	I	T	T	D	I	D	D	D	
J	T	T	J	T	T	T	J	T	T	T	
C	T	C	T	T	C	T	C	C	C	C	...
D	T	T	D	T	D	D	D	D	D	D	
G	T	S	I	J	C	D	G	B	B	B	
B	T	C	D	T	C	D	B	B	B	B	
E	T	C	D	T	C	D	B	B	EE	...	
EE	T	C	D	T	C	D	B	B	...	.....	

Semigroup Table of products of sets in T

T = Full transformation semigroup, S = Surjections, I = Injections, J = top J-class, G = full symmetric group, B = Semi-Balanced elements, C = Collapse majorises Defect, D = Defect majorises Collapse

It follows from the table that $\langle E \cup G \rangle = B$ and we proved that

Theorem

[3, Theorem 5.7] If $\langle E \cup G \cup M \rangle = T_X$ then M contains a proper surjection and a proper injection. Moreover there is a solution to this equation where $|M| = 2$.

In [4] we solved more rank problems where we defined the (*relative*) *rank* of a semigroup S modulo a set $A \subseteq S$ to be the least cardinal r such that $r = |B|$ where $B \subseteq S$ such that $\langle A \cup B \rangle = S$. This is denoted by $\operatorname{rank}(S : A)$. We proved:

Theorem

[4, Theorems 4.1 and 5.1] For an infinite base set X, $rank(T_X : S_X) = rank(T_X : E_X) = 2$.

In [3] we included descriptions of products of E with arbitrary $\mathscr{H}$- $\mathscr{R}$- and $\mathscr{L}$- classes of T_X. An $\mathscr{L}$-class $L = L_\alpha$, is determined by $Y \subseteq X$, the common range of members of L; an $\mathscr{R}$-class R_α is determined by a common partition $\Pi = \mathrm{Ker}\alpha$ of X, and so the $\mathscr{H}$-class $H = L \cap R$ is determined by (Π, Y) so that $H = \{\alpha \in T_n : \mathrm{Ker}\ \alpha = \Pi \text{ and } X\alpha = Y\}$. (Recall that $\mathscr{D} = \mathscr{J}$ in T_X with $(\alpha, \beta) \in \mathscr{D}$ if and only if $|X\alpha| = |X\beta|$.)

Definitions

The Y-kernel of α is $\mathrm{Ker}_Y(\alpha) = \{K \in \mathrm{Ker}\ \alpha : K\alpha \in Y\}$. A Y-transversal of α is a transversal of $\mathrm{Ker}_Y\alpha$. The *relative defect of α is* $D_Y(\alpha) = Y \cap \overline{X\alpha}$ with cardinal $d_Y(\alpha)$. The *relative collapse* of α with respect to Y is the cardinal $c_Y(\alpha) = |X \setminus \tau|$, where τ is some Y-transversal. The *supplement* of a partition Π of X with respect to $\tau \subseteq X$ is denoted by $s_\tau(\Pi)$ and is the cardinal of the set

$$S_\tau(\Pi) = \{P \in \Pi : P \cap \tau = \emptyset\}.$$

The *excess* of α with respect to Y, denoted by $e_Y(\alpha) = |X\alpha \cap \overline{Y}|$.

Definitions

Let $\alpha \in T_X$, $Y \subseteq X$ and Π be a partition of X.
Condition C: there exists a Y-transversal τ of α that is a partial transversal of Π and is such that $s_\tau(\Pi) = d_Y(\alpha)$.

Theorem

(from Thm 2.1 of [3])
(a) $\alpha \in EH$ *iff* $X\alpha \subseteq Y$ *and* α, Y *and* Π *satisfy (C).*
(b) $\alpha \in HE$ *iff* $\mathrm{Ker}\ \alpha \leq \Pi$ *and* α, Y *and* Π *satisfy (C).*
(Thms 2.5, 2.7 of [3])
(a) $\alpha \in LE$ *iff* $c_Y(\alpha) \geq d_Y(\alpha) \geq e_Y(\alpha)$; $\alpha \in EL$ *if additionally* $e_Y(\alpha) = 0$.
(b) $\alpha \in ER$ *iff* $s_\tau(\Pi) \leq d(\alpha)$ *for some transversal* τ *of* $\mathrm{Ker}\ \alpha$ *that is a partial transversal of* Π; $\alpha \in RE$ *if additionally* $\mathrm{Ker}\ \alpha \leq \Pi$.

Theorem

(Thm. 4.1 of [2]) Let L (resp. R) be an $\mathscr{L}$-class (resp. $\mathscr{R}$-class) contained in J other than S (resp. other than I). Then L (resp. R) contains an $\mathscr{H}$-class H such that none of the sets HE, LE, EH, EL, (resp. EH, ER, HE, and RE) are subsemigroups of T_X.

Definitions

(See [1]) Let $\leq$ be the *natural partial order* on a semigroup S, whereby $a \leq b$ iff there exists $s, t \in S^1$ such that $sb = sa = a = at = bt$; if S is regular we may take $s, t \in E(S)$. Let $S_Y = \{\alpha \in T_X : X\alpha \subseteq Y\}$ and S_Π consists of all α such that Ker α has a transversal τ that is a partial transversal of Π; S_Y and S_Π are respectively a left and a right ideal of T_X.

Theorem

Let $L = L_\alpha$ be an $\mathscr{L}$-class and R_α be an $\mathscr{R}$-class, both not contained in J. Then
(a) $EL = L_\downarrow = T\alpha \subset LE = S_Y$;
(b) $RE = R_\downarrow = \alpha T \subset ER = S_\Pi$.

Corollary

Let X be finite. Then
(a) $HE = RE = \alpha T$;
(b) $EH = S_Y \cap S_\Pi$ is a subsemigroup of T and a union of $\mathscr{H}$-classes of T_X.

In general, $EH \subseteq S_Y \cap S_\Pi$ and conversely, if $\alpha \in S_Y \cap S_\Pi$ then for any transversal τ of Kerα that is a partial transversal of Π:

$$|\tau| + s_\tau(\Pi) = |X\alpha| + d_Y(\alpha)$$

since $|\tau| = |X\alpha|$, in the finite case we have simply by subtraction that $s_\tau(\Pi) = d_Y(\alpha)$, which ensures that $\alpha \in EH$ by Condition C. Which $\mathscr{H}$-classes comprise $S = EH$?

Theorem

[Thm. 2.2 of [2]] Let H_1 be defined by (Y_1, Π_1) with $|Y_1| = t$. Then $H_1 \subseteq EH$ iff $Y_1 \subseteq Y$ and Π_1 satisfies Hall's Condition with respect to Π.

Hall's Condition: no union of k classes of Π contains more than k classes of Π_1 $(1 \leq k \leq t-1)$.

Theorem

[Thm. 2.6 of [2]] Let $\alpha, \beta \in S = EH$ be distinct mappings. Then
(a) $L_\beta \leq L_\alpha$ iff $X\beta \subseteq Y\alpha$; $\alpha \mathscr{L} \beta$ iff $X\alpha = Y\alpha = X\beta = Y\beta$.
(b) $R_\beta \leq R_\alpha$ iff Ker $\beta \leq$Ker α; $\alpha \mathscr{R} \beta$ iff Ker $\alpha =$ Ker β.
(c) $\alpha \in Reg(S)$ iff $X\alpha = Y\alpha$.

Definitions

A partition Γ of X that satisfies Hall's Condition wrt Π is called *regular* wrt $S = EH$ if $K \cap Y \neq \emptyset$ for every $K \in \Gamma$. Let S_R (resp. S_I) denote the set of all regular (resp. irregular) partitions of X wrt S.

Theorem

For each $\Gamma \in S_I$ there is an irregular $\mathscr{D}$-class D_Γ whereby $\alpha \in D_\Gamma$ iff $Ker\ \alpha = \Gamma$ and $X\alpha \subseteq Y$ with $|X\alpha| = |\Gamma|$. In particular, each irregular $\mathscr{D}$-class D is also an $\mathscr{R}$-class and each $\mathscr{L}$-class within D is trivial. If S is not regular (occurs when $|Y| \geq 2$ and $H \neq S_X$) then there exists irregular members of S of all ranks $2 \leq m \leq |Y|$. There are $|Y|$ (non-empty) regular $\mathscr{D}$-classes D_m, $(1 \leq m \leq |Y|)$;

$$D_m = \{\alpha : |X\alpha| = m, X\alpha \subseteq Y, Ker\ \alpha \text{ a regular partition of } X \text{ wrt } S\}.$$

Furthermore $Reg(S)$ is a right ideal of S.

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