E-unitary almost factorisable orthodox semigroups

Miklós Hartmann

University of York

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Semidirect products

Semidirect products

Let G be a group acting on the band B from the left. Then the semidirect product of B by G is the set $B \times G$ endowed with the following multiplication:

$$(e, g)(f, h) = (e \cdot {}^{g}f, gh).$$

The semidirect product is denoted by $B \rtimes G$.

A natural question

How inverse semigroups relate to semidirect products of semilattices by groups?

Route I: homomorphic images of subsemigroups

E-unitary orthodox semigroups

A regular semigroup S is E-unitary if for all $e \in E(S)$ and $a \in S$, both $ae \in E(S)$ and $ea \in E(S)$ imply $a \in E(S)$.
If S is inverse, the definition is simplified: $e \in E(S),\ e \leq a$ implies $a \in E(S)$.

McAlister, 1974

Every inverse semigroup is an (idempotent separating) homomorphic image of an E-unitary inverse semigroup.

O'Carroll, 1976

Every E-unitary inverse semigroup is embeddable into a semidirect product.

Route II: subsemigroups of homomorphic images - the monoid case

Factorisable monoids

An inverse monoid M is factorisable if for every $m \in M$ there exist $e \in E(M)$ and $u \in U(M)$ such that $m = eu$.

Chen, Hsieh, 1974

Every inverse semigroup is embeddable into a factorisable inverse monoid.

The monoid case

Factorisable inverse monoids are exactly the (idempotent separating) monoid homomorphic images of semidirect products of semilattices (semilattice monoids) by groups.

Proof

The group $U(M)$ acts on $E(M)$ by

$$^{g}e = geg^{-1},$$

and the mapping

$$\omega\colon E(M) \rtimes U(M) \to M, (e,u) \mapsto eu$$

is a surjective homomorphism.

Route II: the semigroup case

The translational hull

Let S be a semigroup. Then the right map $\rho\colon S \to S$ is a right translation of S if $(st)\rho = s(t\rho)$ for all $s, t \in S$. A left translation is defined dually. If λ is a left translation and ρ is a right translation then (λ, ρ) is a linked pair if $(s\rho)t = s(\lambda t)$ for every $s, t \in S$. Let

$$\Omega(S) = \{(\lambda, \rho) : (\lambda, \rho) \text{ is a linked pair}\} \subseteq \mathcal{T}(S) \times \mathcal{T}^*(S).$$

Then $\Omega(S)$ is a semigroup, called the translational hull of S.

Fortunately we are only interested in its group of units...

$\Sigma(S)$

We denote the group of units of $\Omega(S)$ by $\Sigma(S)$.

Almost factorisable inverse semigroups

Definition

The inverse semigroup S is almost factorisable if for every $s \in S$ there exist $e \in E(S)$ and $(\lambda, \rho) \in \Sigma(S)$ such that $s = e\rho$.

McAlister, 1976, Lawson, 1994

Almost factorisable inverse semigroups are exactly the (idempotent separating) homomorphic images of semidirect products.

Pro:

The group $\Sigma(S)$ acts on $E(S)$ by the rule ${}^{(\lambda,\rho)}e = \lambda e \rho^{-1}$. The mapping

$$\omega\colon E(S) \rtimes \Sigma(S) \to S, (e, (\lambda, \rho)) = e\rho$$

is a surjective idempotent separating homomorphism.

The orthodox case - E-unitary orthodox semigroups

Takizawa, 1979, Szendrei, 1980

Every orthodox semigroup is an idempotent separating homomorphic image of an E-unitary orthodox semigroup.

Szendrei, 1993

Every orthodox semigroup has an E-unitary cover which is embeddable into a semidirect product.

Billhardt, 1998

But not every E-unitary orthodox semigroup is embeddable into a semidirect product.

The orthodox case - (almost) factorisable orthodox monoids (semigroups)

Just turn back to the pages containing the inverse case, replace 'inverse' by 'orthodox', 'semilattice' by 'band' and in the semigroup case delete the brackets.

In the monoid case the trick works even for the proofs... Except for Chen and Hsieh's result.

The orthodox case - factorisable orthodox monoids

Factorisable monoids

An orthodox monoid M is factorisable if for every $m \in M$ there exist $e \in E(M)$ and $u \in U(M)$ such that $m = eu$.

Factorisable orthodox monoids are exactly the (idempotent separating) monoid homomorphic images of semidirect products of bands (band monoids) by groups.

Hartmann, 2007

Every orthodox semigroup is embeddable into a factorisable orthodox monoid. (This result is much harder to prove than in the inverse case).

The orthodox case - almost factorisable orthodox semigroups

Definition

The orthodox semigroup S is almost factorisable if for every $s \in S$ there exist $e \in E(S)$ and $(\lambda, \rho) \in \Sigma(S)$ such that $s = e\rho$.

Hartmann, 2007

Almost factorisable orthodox semigroups are exactly the idempotent separating homomorphic images of semidirect products.

E-unitary factorisable orthodox monoids

Claim

An orthodox monoid is E-unitary and factorisable if and only if it is isomorphic to a semidirect product.

Proof:

If M is factorisable then the mapping

$$\omega\colon E(M) \rtimes U(M) \to M, (e, u) \mapsto eu$$

is a surjective homomorphism. Furthermore,
$eu = fv \iff e = f \cdot (vu^{-1})$, so that vu^{-1} is idempotent, that is,
$u = v$ and $e = f$.
So ω is an isomorphism.

E-unitary almost factorisable inverse semigroups

Theorem

An inverse semigroup is E-unitary and almost factorisable if and only if it is isomorphic to a semidirect product.

Proof:

The mapping

$$\omega\colon E(S) \rtimes \Sigma(S) \to S, (e, (\lambda, \rho)) \mapsto e\rho$$

is a surjective homomorphism.
Suppose first that $e\tau \in E(S)$. We have that $e\tau \mathcal{R} e$, so $e\tau = e$. If $e\rho = e'\rho'$, then $e = e'(\rho'\rho^{-1})$, so $e = e'$. Denote $\rho'\rho^{-1}$ by τ.
Let $f \in E(S)$. Then $e \cdot (f\tau)^{-1} = e \cdot \nu^{-1} f = e\tau^{-1} \cdot f =$
$= e \cdot f \in E(S)$, where $(\nu, \tau) \in \Sigma(S)$. Thus $f\tau \in E(S)$, which implies $f\tau = f$. Which implies that τ is the identity map. Which in turn implies that $\rho = \rho'$.

E-unitary almost factorisable orthodox semigroups

Connection between semigroups and monoids - Lawson, 1994, Hartmann, 2007

If M is a factorisable orthodox monoid, then $M \setminus U(M)$ is an almost factorisable orthodox semigroup.
Conversely, if S is an almost factorisable orthodox semigroup, then $S \cup \Sigma(S)$ is a factorisable orhodox monoid.
It is easy to see that if $S \cup G$ is isomorphic to a semidirect product for some $G \leq \Sigma(S)$, then S is such, too.

An idea

If S is an E-unitary almost factorisable orthodox semigroup, then $S \cup \Sigma(S)$ is a factorisable orthodox monoid. Under what conditions will it be E-unitary? Can we choose a subgroup $G \leq \Sigma(S)$ such that $S \cup G$ is still factorisable, and even E-unitary?

Making use of the greatest inverse semigroup homomorphic image

Connection between $\Sigma(S)$ and $\Sigma(S/\gamma)$

Let $(\lambda, \rho) \in \Sigma(S)$. Then the mapping

$$\rho_\gamma \colon S/\gamma \to S/\gamma, (s\gamma)\rho_\gamma \mapsto (s\rho)\gamma$$

is a well-defined right translation of S/γ. Dually we can define λ_γ. The mapping

$$\chi \colon \Sigma(S) \to \Sigma(S/\gamma), (\lambda, \rho) \mapsto (\lambda_\gamma, \rho_\gamma)$$

is a homomorphism.

A lemma

Lemma

If S is an orthodox semigroup and $G \leq \Sigma(S)$, then $S \cup G$ is factorisable if and only if $S\gamma \cup G\chi$ is factorisable.

Proof:

Let $s \in S$ and let $e \in E(S)$, $(\lambda, \rho) \in G$ be such that $s\gamma = e\gamma\rho_\gamma$. Then

$$s = (ss' \cdot \lambda(s's)\rho^{-1})\rho$$

Note that if $s = e\rho$, then

$$e \,\mathcal{R}\, s\rho \,\mathcal{L}\, \lambda^{-1}e\rho$$

Conditions on $G \leq \Sigma(S)$ - S is E-unitary almost factorisable

Condition I.

It is easy to see that $M = S \cup G$ is factorisable if and only if every σ-class of M contains an element of G.

Condition II.

In order to ensure that $S \cup G$ is still E-unitary, we need that every σ-class of M contains at most one element of G, because G is just the $\mathcal{H}$-class of 1.

Summary

That is, an E-unitary almost factorisable orthodox semigroup S is isomorphic to a semidirect product if and only if $\Sigma(S)$ has a subgroup G such that $G\chi = \Sigma(S/\gamma) \cong S/\sigma$.

We do not like the translational hull...

In general, the translational hull of an orthodox semigroup S is very complicated, but its group of units has some connections with the automorphism group of its band.
Namely, $\Sigma(S)$ acts on $E(S)$ by the rule

$$^{(\lambda,\rho)}e = \lambda e \rho^{-1}.$$

That is, if we have a homomorphism from a group G to $\Sigma(S)$, this determines an action of G on $E(S)$.

Under some conditions, the converse is true: an action of G on $E(S)$ determines a homomorphism from G to $\Sigma(S)$. One of these conditions is that the action of G on B must be compatible with the action of $\Sigma(S/\gamma)$ on $E(S)/\mathcal{D}$.

Generalised inverse semigroups

Definition

An orthodox semigroup is a generalised inverse semigroup if it has a normal band of idempotents.

In case of generalised inverse semigroups, the aforementioned condition is the only one. To be able to formulate it properly, let S be an E-unitary almost factorisable generalised inverse semigroup. Then S/γ is isomorphic to a semidirect product, thus S/σ acts on the semilattice $E(S)/\mathcal{D}$. We fix this action.

Theorem

An E-unitary almost factorisable generalised inverse semigroup S is isomorphic to a semidirect product if and only if there exists an action of S/σ on $E(S)$ such that for every $e \in E(S)$ and $g \in S/\sigma$,

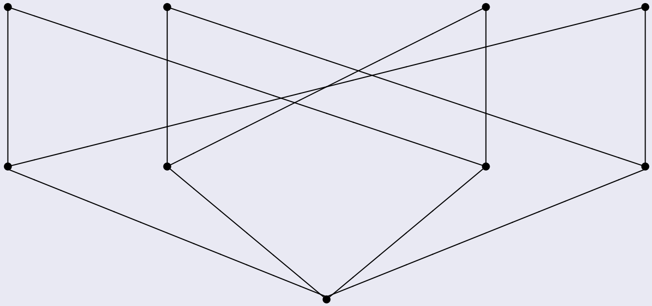
$$({}^{g}e)\mathcal{D} = {}^{g}(e\mathcal{D}).$$

Much ado for nothing? Fortunately not.

Theorem

There exists an E-unitary almost factorisable generalised inverse semigroup which is not isomorphic to a semidirect product.

The band's structure semilattice



E-unitary homomorphic images of semidirect products

Proposition

Let $B \rtimes G$ be a semidirect product and let $N \lhd G$ be such that for every $a, b \in B$ and $n \in N$,

$$ {}^{n}(ab) \cdot b \,\mathcal{L}\, ab \,\mathcal{R}\, a \cdot {}^{n}(ab). $$

Then the relation κ defined by

$$ (a, g) \,\kappa\, (b, h) \Longleftrightarrow Ng = Nh, a \,\mathcal{R}\, b \text{ and } {}^{g^{-1}}a \,\mathcal{L}\, {}^{g^{-1}}b $$

is an idempotent separating E-unitary congruence on $B \rtimes G$. Conversely, each idempotent separating E-unitary congruence κ on the semidirect product $B \rtimes G$ is of this form for a unique normal subgroup N in G.

Construction of E-unitary almost factorisable orthodox semigroups

Theorem, Hartmann, Szendrei, 2012

Let B be a band and H be a group. Consider a mapping $\xi\colon H \to \mathrm{Aut}^d\ B,\ h \mapsto \xi_h$ such that the following conditions are satisfied by ξ and by the mapping

$$\eta\colon H \times H \to \mathrm{Aut}^d\ B, (h,k) \mapsto \eta_{h,k} = \xi_{hk}\xi_k^{-1}\xi_h^{-1} :$$

- $\xi_1 = \mathrm{id}_B$,
- $a\, \mathcal{D}\, \eta_{h,k}a$ for every $h,k \in H$ and $a \in B$,
- $\eta_{h,k}(ab)\cdot a\, \mathcal{L}\, ab\, \mathcal{R}\, a\cdot\eta_{h,k}(ab)$ for every $h,k \in H$ and $a,b \in B$.

Continued...

Then the set $B \times H$ endowed with the multiplication

$$(a, h)(b, k) = (a \cdot \xi_h b \cdot \eta_{h,k}(a \cdot \xi_h b), hk)$$

is an E-unitary almost factorisable orthodox semigroup, called the Q-product of B by H.
Conversely, each E-unitary almost factorisable orthodox semigroup is isomorphic to a Q-product.