MITSCH'S ORDER AND INCLUSION FOR BINARY RELATIONS

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ABSTRACT. Mitsch's natural partial order on the semigroup of binary relations has a complex relationship with the compatible partial order of inclusion. This relationship is explored by means of a sublattice of the lattice of preorders on the semigroup. The natural partial order is also characterised by equations in the theory of relation algebras.

1. Natural partial orders

Having formulated a characterisation of the natural partial order $\leq$ on the full transformation semigroup $\mathcal{T}_X$ which did not require inverses or idempotents, Mitsch [6] went on to define the natural partial order $\leq$ on any semigroup S by

(1) $\quad a \leq b$ if $a = b$ or there are $x, y \in S$ such that $a = ax = bx = yb$

for $a, b \in S$. Observe that $a = ya$ follows. Mitsch's natural partial order has now been characterised, and its properties investigated, for several concrete classes of non-regular semigroups—in [5, 8] for some semigroups of (partial) transformations, and by Namnak and Preechasilp [7] for the semigroup $\mathcal{B}_X$ of all binary relations on the set X.

The partial order of inclusion which is carried by $\mathcal{B}_X$ may also be thought of as 'natural', and it is the broad purpose of this note to discuss the relationship between these two partial orders on $\mathcal{B}_X$. So we shall use a slightly different nomenclature here for the sake of clarity, mostly referring to partial orders as just *orders,* and the natural partial order as *Mitsch's order*. We first collect some information about $\mathcal{B}_X$.

2. Binary relations

The notation used here for binary relations follows that found in, for example, Clifford and Preston [1], with the addition of complementation of relations defined by

$$x \, \alpha^c \, y \iff (x, y) \notin \alpha$$

for $x, y \in X$. Note that the symbol $\circ$ for composition will be suppressed, except for the composites of order relations on $\mathcal{B}_X$. We will make use of the identity relation on X, $\iota = \{(x, x) : x \in X\}$, and the universal relation $\omega = X \times X$.

The following logical equivalence will also be required; it is the 'Theorem K' of De Morgan [2, p. *xxx*] (see also e.g. [3]).

Result 2.1. *For $\alpha, \beta, \xi \in \mathcal{B}_X$,*

$$\beta\xi \subseteq \alpha \iff \xi \subseteq \left(\beta^{-1}\alpha^c\right)^c \iff \beta \subseteq \left(\alpha^c\xi^{-1}\right)^c.$$

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Result 2.1 will be used in the following form:

Corollary 2.2. *(i) If the set* $\{\xi \in \mathcal{B}_X : \beta\xi = \alpha\}$ *is non-empty, it has greatest element* $(\beta^{-1}\alpha^c)^c$ *in the inclusion order.*
(ii) If the set $\{\xi \in \mathcal{B}_X : \xi\beta = \alpha\}$ *is non-empty, it has greatest element* $(\alpha^c\beta^{-1})^c$ *in the inclusion order.*

Proof. (i) By Result 2.1 above, $\beta\xi = \alpha$ implies $\xi \subseteq (\beta^{-1}\alpha^c)^c$; but the latter implies, again using Result 2.1, that

$$\alpha = \beta\xi \subseteq \beta\left(\beta^{-1}\alpha^c\right)^c \subseteq \alpha.$$

Part (ii) is proven dually. □

3. Equational criterion for the Mitsch order on $\mathcal{B}_X$

The basic definition (1) of $\alpha \le \beta$ is existentially quantified. In $\mathcal{B}_X$ there is a purely equational equivalent:

Theorem 3.1. *For* $\alpha, \beta \in \mathcal{B}_X$, $\alpha \le \beta$ *if and only if*

$$\alpha = \alpha\left(\beta^{-1}\alpha^c\right)^c = \beta\left(\beta^{-1}\alpha^c\right)^c = \left(\alpha^c\beta^{-1}\right)^c\beta.$$

Proof. Suppose $\alpha \le \beta$. By definition there are ξ, η such that

$$\alpha = \alpha\xi = \beta\xi = \eta\beta$$

and hence both

$$\eta\alpha = \eta\beta\xi = \alpha\xi = \alpha$$

and

$$\alpha = \beta\left(\beta^{-1}\alpha^c\right)^c = \left(\alpha^c\beta^{-1}\right)^c\beta,$$

the latter by Corollary 2.2. Now

$$\begin{aligned}\alpha = \eta\alpha = \eta\beta\left(\beta^{-1}\alpha^c\right)^c &= \alpha\left(\beta^{-1}\alpha^c\right)^c \\ &= \beta\left(\beta^{-1}\alpha^c\right)^c = \left(\alpha^c\beta^{-1}\right)^c\beta.\end{aligned}$$

Conversely,

$$\alpha = \alpha\left(\beta^{-1}\alpha^c\right)^c = \beta\left(\beta^{-1}\alpha^c\right)^c = \left(\alpha^c\beta^{-1}\right)^c\beta$$

demonstrates $\alpha \le \beta$. □

Regarded as a computation, this criterion is of polynomial time complexity in $|X|$, as is also the case for tests of the divisibility preorders (by Corollary 2.2, $\alpha = \beta\xi$ if and only if $\alpha = \beta\left(\beta^{-1}\alpha^c\right)^c$, etc.), but in contrast to the NP-complete tests for the $\mathcal{J}$-preorder [4]. Of course algorithmic complexity is not the only issue. Namnak and Preechasilp [7] characterise Mitsch's order for binary relations with the aid of Zaretskiĭ's criteria for divisibility [9] which, though also of worst-case exponential complexity, proved very convenient for the purposes of finding compatible elements, atoms and maximal elements in the Mitsch order [7]. The equations of Theorem 3.1 are complex in the different sense that they belong to a theory of semigroups enriched by operations of inversion and complementation, in fact, to the theory of relation algebras [3]. However, Theorem 3.1 suggested some results found in the later sections of this paper, although the proofs used here are simpler ones based directly on the definition (1). It may be observed that all the proofs apply to (abstract) relation algebras [3] and not just the representable ones $\mathcal{B}_X$.

4. Connexions with the inclusion order

In discussing Mitsch's order and the inclusion order on $\mathcal{B}_X$, [7] notes their logical independence. In fact, we can see that every inclusion atom is a Mitsch atom, and every non-empty relation on X is in an inclusion interval between two Mitsch atoms. Moreover, permutations of X are Mitsch-maximal, but far from being either maximal elements or atoms in the inclusion order. Yet the relationship between the two orders is subtle, and worthy of further exploration. We illustrate this by next finding a substructure of $\mathcal{B}_X$ in which $\leq$ agrees with $\subseteq$, and others where $\leq$ agrees with reverse inclusion $\supseteq$. We shall use the following statement, easily proved.

Lemma 4.1. *Let S be a semigroup, T be a regular subsemigroup of S, and $a, b \in T$. Then $a \leq b$ in T if and only if $a \leq b$ in S.*

The symmetric inverse monoid $\mathcal{I}_X$ is a regular subsemigroup of $\mathcal{B}_X$, and the natural partial order on $\mathcal{I}_X$ coincides with inclusion. So we may apply Lemma 4.1 to $\mathcal{I}_X$.

Corollary 4.2. *(i) If $\alpha, \beta \in \mathcal{I}_X$, then $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$.*
(ii) If $\beta \in \mathcal{I}_X$, then $\alpha \subseteq \beta$ implies $\alpha \leq \beta$.

Part (ii) relies on the observation that $\beta \in \mathcal{I}_X$ and $\alpha \subseteq \beta$ imply $\alpha \in \mathcal{I}_X$. Of course, $\alpha \leq \beta$ need not entail $\alpha \in \mathcal{I}_X$.

To obtain pairs (α, β) such that $\leq$ agrees with reverse inclusion $\supseteq$, first let α be a reflexive and transitive relation on X (so, a preorder), and define a subset of $\mathcal{B}_X$ by

$$F(\alpha) = \{\beta \in \mathcal{B}_X : \alpha\beta = \alpha = \beta\alpha\}.$$

Proposition 4.3. *For all $\beta \in F(\alpha)$, $\alpha \leq \beta$ if and only if $\beta \subseteq \alpha$.*

Proof. Clearly $F(\alpha)$ is a subsemigroup of $\mathcal{B}_X$, with zero element α. So for all $\beta \in F(\alpha)$, we have $\alpha \leq \beta$ and, since $\iota \subseteq \alpha$, also $\beta \subseteq \beta\alpha = \alpha$. □

Corollary 4.2 shows that it may be instructive to consider the conjunction of the natural partial order with set inclusion, which is an order on $\mathcal{B}_X$ which we naturally write as $\leq \cap \subseteq$. Similarly, Proposition 4.3 suggests investigating the conjuction of $\leq$ with reverse inclusion, an order written as $\leq \cap \supseteq$. The next Proposition, which extends Proposition 4.3, gives an alternative characterisation for $\leq \cap \supseteq$; there seems to be no analogous description of $\leq \cap \subseteq$.

Proposition 4.4. *For all $\alpha, \beta \in \mathcal{B}_X$, $\alpha \leq \beta$ and $\alpha \supseteq \beta$ if and only if there are $\varepsilon = \varepsilon^2$ and $\phi = \phi^2$ such that $\iota \subseteq \varepsilon$ and $\alpha = \beta\varepsilon = \phi\beta$.*

Proof. Let $\alpha \leq \beta$. From Corollary 2.2 there exists the element ε, maximum with respect to $\subseteq$ such that $\alpha = \beta\varepsilon$, and $\alpha = \alpha\varepsilon$ also holds by Theorem 3.1. Then also $\alpha = \beta\varepsilon^2$ and thus $\varepsilon^2 \subseteq \varepsilon$. But from $\beta\iota \subseteq \alpha$ we have $\iota \subseteq \varepsilon$, so $\varepsilon \subseteq \varepsilon^2$ and $\varepsilon = \varepsilon^2$. Similarly $\phi = \phi^2$ with $\alpha = \phi\beta$.

Conversely, if the conditions hold then plainly $\alpha \leq \beta$ as in the regular case, but also $\iota \subseteq \varepsilon$ implies $\beta \subseteq \varepsilon\beta = \alpha$. □

5. The sublattice of preorders generated by $\leq$, $\subseteq$ and $\supseteq$

The subsemigroups $F(\alpha)$ and $\mathcal{I}_X$ of section 4 clearly show that, if $|X| \geq 2$, there can be no order relation on $\mathcal{B}_X$ which contains $\leq$ and either of $\subseteq$ and $\supseteq$. However the set of preorders on $\mathcal{B}_X$ is bounded by the relation of equality $=$ and the universal relation on $\mathcal{B}_X$, and is closed under arbitrary intersections, and so forms a lattice. In a similar situation, the papers [5] and [8] derive interesting results from considering the composite of inclusion with $\leq$, and this idea also turns out to be useful here.

Proposition 5.1. *For all $\alpha, \beta \in \mathcal{B}_X$, there exists $\gamma \in \mathcal{B}_X$ such that $\alpha \subseteq \gamma \leq \beta$ if and only if $\alpha\omega\alpha \subseteq \beta\omega\beta$.*

Proof. Suppose $\alpha \subseteq \gamma \leq \beta$. Then there are ξ, η such that $\gamma = \beta\xi = \eta\beta$, so

$$\alpha\omega\alpha \subseteq \gamma\omega\gamma = \beta\xi\omega\eta\beta \subseteq \beta\omega\beta.$$

Since $\beta \subseteq \beta\beta^{-1}\beta \subseteq \beta\omega\beta$, we have

$$\beta\omega\beta \subseteq \beta\omega\beta\omega\beta \subseteq \beta\omega\beta.$$

But now $\beta\omega\beta = \beta\omega\beta\omega\beta$ shows that $\beta\omega\beta \leq \beta$. As above we have $\alpha \subseteq \alpha\omega\alpha$, so $\alpha\omega\alpha \subseteq \beta\omega\beta$ implies $\alpha \subseteq \beta\omega\beta \leq \beta$. □

Remark 5.2. *The relation of two-sided divisibility $\leq_H$, defined on $\mathcal{B}_X$ by*

$$\alpha \leq_H \beta \text{ if there are } \xi, \eta \in \mathcal{B}_X \text{ such that } \alpha = \beta\xi = \eta\beta,$$

is a preorder containing $\leq$. The proof of Proposition 5.1 also shows that $\subseteq \circ \leq_H = \subseteq \circ \leq$.

As a corollary, we have the join of the Mitsch and inclusion orders in the lattice of preorders on $\mathcal{B}_X$.

Corollary 5.3. *(i) The composite $\leq \circ \subseteq$ is contained in $\subseteq \circ \leq$.*
(ii) $\subseteq \circ \leq$ is the join of $\subseteq$ and $\leq$ in the lattice of preorders on $\mathcal{B}_X$.

Proof. (i) First let us note that $\leq$ and $\subseteq$ are subsets of $\subseteq \circ \leq$. It is clear from Proposition 5.1 that $\subseteq \circ \leq$ is transitive, i.e.,

$$(\subseteq \circ \leq) \circ (\subseteq \circ \leq) = (\subseteq \circ \leq)$$

and it follows that $(\leq \circ \subseteq)$ is contained in $(\subseteq \circ \leq)$.

(ii) Also it is immediate that $\subseteq \circ \leq$ is reflexive, and so it is a preorder on $\mathcal{B}_X$. Any preorder on $\mathcal{B}_X$ containing both $\subseteq$ and $\leq$ also contains $\subseteq \circ \leq$. Hence $\subseteq \circ \leq$ is the join of $\subseteq$ and $\leq$. □

That the containment in (i) is proper (for $|X| \geq 2$) is shown by the following instance. Consider a pair of distinct permutations π, ρ: we have $\pi \subseteq \omega \leq \rho$, but $\pi \leq \alpha \subseteq \rho$ implies $\pi = \alpha = \rho$, a contradiction. It also follows that $\leq \circ \subseteq$ is not a preorder.

We turn to the composites and join of $\leq$ with reverse inclusion.

Proposition 5.4. *The composite $\supseteq \circ \leq$ is the universal relation on $\mathcal{B}_X$ and the join of $\supseteq$ and $\leq$ in the lattice of preorders on $\mathcal{B}_X$.*

Proof. For any $\alpha, \beta \in \mathcal{B}_X$, $\alpha \supseteq \varnothing \leq \beta$, so $\mathcal{B}_X \times \mathcal{B}_X$ coincides with $\supseteq \circ \leq$. Now $\mathcal{B}_X \times \mathcal{B}_X$ is plainly a preorder, and any preorder containing both $\leq$ and $\supseteq$ must contain $\supseteq \circ \leq$ and hence $\mathcal{B}_X \times \mathcal{B}_X$. □

Here too, we see that the reverse composite $\leq \circ \supseteq$ is properly contained in $\supseteq \circ \leq = \mathcal{B}_X \times \mathcal{B}_X$ when $|X| \geq 2$ (and so is not a preorder), since $\iota \leq \alpha \supseteq \omega$ implies both $\alpha = \iota$ and $\alpha = \omega$.

Thus we are able to describe the relationships between $\leq$, $\subseteq$ and $\supseteq$ in terms of the sublattice generated by $\leq, \subseteq$ and $\supseteq$ within the lattice of preorders on $\mathcal{B}_X$. This sublattice is summarised by a Hasse diagram in Fig. 1; filled circles denote orders, and additional labels in parentheses summarise conditions for α, β to be related by the preorder.

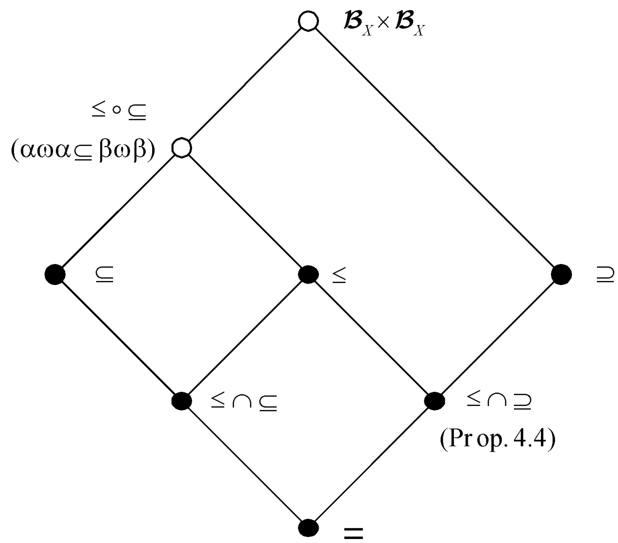

Fig. 1. A sublattice of preorders on B_X. Filled circles denote orders.

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