Weakly U-abundant semigroups

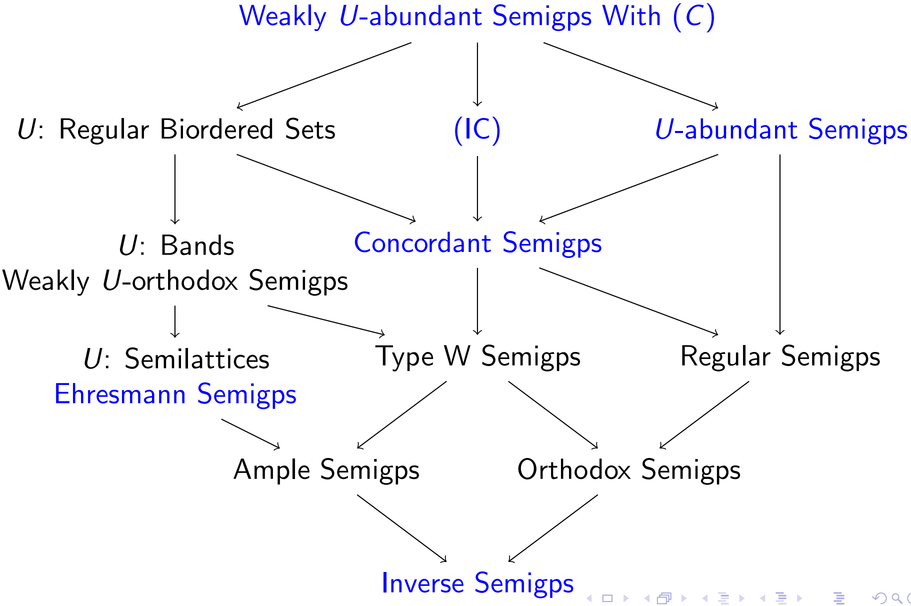
Yanhui Wang

The University of York, UK

23rd Nov. 2011

NBSAN

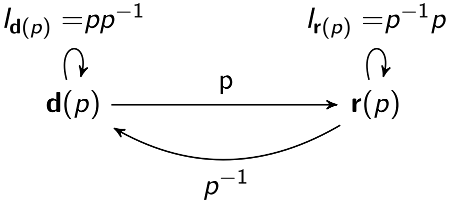
Outline: U a biordered set



Groupoids

Definition A *groupoid* is a category G in which for every $p \in \mathrm{Mor}\, G$ we have $p^{-1} \in \mathrm{Mor}\, G$ with

$$pp^{-1} = I_{\mathbf{d}(p)} \text{ and } p^{-1}p = I_{\mathbf{r}(p)}.$$



Inverse semigroups → Groupoids

Let S be an inverse semigroup with semilattice of idempotents E. We construct a groupoid $C(S)$ as follows:

$$\text{Ob } C(S) = E, \ \text{Mor}\, C(S) = S, \ \mathbf{d}(x) = xx', \mathbf{r}(x) = x'x$$

and a partial binary operation $\cdot$ is defined by the rule that for any $x, y \in S$,

$$x \cdot y = \begin{cases} xy & \text{if } \ \mathbf{r}(x) = \mathbf{d}(y), \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

where xy is the product of x and y in S.

Groupoids $\rightarrow$ Inverse semigroups

Let $G = (G, \cdot)$ be a small groupoid. Let S be the semigroup obtained from G by declaring all undefined products to be 0.

Fact 1. $S = G \cup \{0\}$ is an inverse semigroup with 0 with $E(S) = E(G) \cup \{0\}$.

Groupoids $\rightarrow$ Inverse semigroups

Let $G = (G, \cdot)$ be a small groupoid. Let S be the semigroup obtained from G by declaring all undefined products to be 0.

Fact 1. $S = G \cup \{0\}$ is an inverse semigroup with 0 with $E(S) = E(G) \cup \{0\}$.

Inductive groupoids

Let $G = (G, \cdot)$ be a small groupoid with $E(G) = E$. Suppose that $\leq$ is a partial order on G. Suppose also

1. $x \leq y$ implies that $x^{-1} \leq y^{-1}$;
2. $x \leq y, u \leq v, \exists x \cdot u, \exists y \cdot v$ implies that $x \cdot u \leq y \cdot v$;
3. if $a \in G$ and $e \in E$ with $e \leq \mathbf{d}(a)$, then there exists a unique **restriction** $(e|a) \in G$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;
4. if $a \in G$ and $e \in E$ with $e \leq \mathbf{r}(a)$, then there exists a unique **co-restriction** $(a|e) \in G$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$;
   then $G = (G, \cdot, \leq)$ is called an **ordered groupoid**. If in addition
5. E is a semilattice.

Then $G = (G, \cdot, \leq)$ is called an **inductive groupoid**.

Inductive groupoids

Let $G = (G, \cdot)$ be a small groupoid with $E(G) = E$. Suppose that $\leq$ is a partial order on G. Suppose also

1. $x \leq y$ implies that $x^{-1} \leq y^{-1}$;
2. $x \leq y, u \leq v, \exists x \cdot u, \exists y \cdot v$ implies that $x \cdot u \leq y \cdot v$;
3. if $a \in G$ and $e \in E$ with $e \leq \mathbf{d}(a)$, then there exists a unique **restriction** $(e|a) \in G$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;
4. if $a \in G$ and $e \in E$ with $e \leq \mathbf{r}(a)$, then there exists a unique **co-restriction** $(a|e) \in G$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$;
   then $G = (G, \cdot, \leq)$ is called an **ordered groupoid**. If in addition
5. E is a semilattice.

Then $G = (G, \cdot, \leq)$ is called an **inductive groupoid**.

Inductive groupoids

Let $G = (G, \cdot)$ be a small groupoid with $E(G) = E$. Suppose that $\leq$ is a partial order on G. Suppose also

1. $x \leq y$ implies that $x^{-1} \leq y^{-1}$;
2. $x \leq y, u \leq v, \exists x \cdot u, \exists y \cdot v$ implies that $x \cdot u \leq y \cdot v$;
3. if $a \in G$ and $e \in E$ with $e \leq \mathbf{d}(a)$, then there exists a unique **restriction** $(e|a) \in G$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;
4. if $a \in G$ and $e \in E$ with $e \leq \mathbf{r}(a)$, then there exists a unique **co-restriction** $(a|e) \in G$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$;
   then $G = (G, \cdot, \leq)$ is called an **ordered groupoid**. If in addition
5. E is a semilattice.

Then $G = (G, \cdot, \leq)$ is called an **inductive groupoid**.

Inductive groupoids

Let $G = (G, \cdot)$ be a small groupoid with $E(G) = E$. Suppose that $\leq$ is a partial order on G. Suppose also

1. $x \leq y$ implies that $x^{-1} \leq y^{-1}$;
2. $x \leq y, u \leq v, \exists x \cdot u, \exists y \cdot v$ implies that $x \cdot u \leq y \cdot v$;
3. if $a \in G$ and $e \in E$ with $e \leq \mathbf{d}(a)$, then there exists a unique **restriction** $(e|a) \in G$ with $\mathbf{d}(e|a) = e$ and $(e|a) \leq a$;
4. if $a \in G$ and $e \in E$ with $e \leq \mathbf{r}(a)$, then there exists a unique **co-restriction** $(a|e) \in G$ with $\mathbf{r}(a|e) = e$ and $(a|e) \leq a$;
   then $G = (G, \cdot, \leq)$ is called an **ordered groupoid**. If in addition
5. E is a semilattice.

Then $G = (G, \cdot, \leq)$ is called an **inductive groupoid**.

Inverse semigroups $\rightarrow$ Inductive groupoids

Let S be an inverse semigroup with semilattice of idempotents E. Let $C(S) = (S, \cdot)$ be defined as above.

Partial Order: For any $a, b \in S$, $a \leq b \;\Leftrightarrow\; a = eb$ for some $e \in E$

Fact 2. $C(S) = (S, \cdot, \leq)$ is an inductive groupoid with

$$(e|a) = ea, \; (a|e) = ae.$$

Inverse semigroups → Inductive groupoids

Let S be an inverse semigroup with semilattice of idempotents E. Let $C(S) = (S, \cdot)$ be defined as above.

Partial Order: For any $a, b \in S$, $a \leq b \Leftrightarrow a = eb$ for some $e \in E$

Fact 2. $C(S) = (S, \cdot, \leq)$ is an inductive groupoid with

$$(e|a) = ea, \ (a|e) = ae.$$

Let S be an inverse semigroup with semilattice of idempotents E. Let $C(S) = (S, \cdot)$ be defined as above.

Partial Order: For any $a, b \in S$, $a \leq b \Leftrightarrow a = eb$ for some $e \in E$

Fact 2. $C(S) = (S, \cdot, \leq)$ is an inductive groupoid with

$$(e|a) = ea, \ (a|e) = ae.$$

Inductive groupoids → Inverse semigroups

Let $G = (G, \cdot, \leq)$ be an inductive groupoid.
We define a *pseudo-product* $\otimes$ on G by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Fact 3. $\mathcal{S}(G) = (G, \otimes)$ is an inverse semigroup (having the same partial order as G).

Ehresmann-Schein-Nambooripad
The category $\mathbf{I}$ of inverse semigroups and morphisms is isomorphic to the category $\mathbf{G}$ of inductive groupoids and inductive functors.
◀

Inductive groupoids → Inverse semigroups

Let $G = (G, \cdot, \leq)$ be an inductive groupoid.
We define a *pseudo-product* $\otimes$ on G by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Fact 3. $\mathcal{S}(G) = (G, \otimes)$ is an inverse semigroup (having the same partial order as G).

Ehresmann-Schein-Nambooripad
The category **I** of inverse semigroups and morphisms is isomorphic to the category **G** of inductive groupoids and inductive functors.
◀

Inductive groupoids $\rightarrow$ Inverse semigroups

Let $G = (G, \cdot, \leq)$ be an inductive groupoid.
We define a *pseudo-product* $\otimes$ on G by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

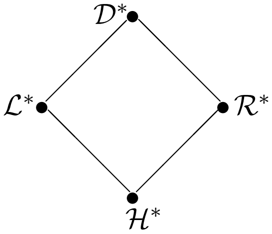
Fact 3. $\mathcal{S}(G) = (G, \otimes)$ is an inverse semigroup (having the same partial order as G).

Ehresmann-Schein-Nambooripad
The category **I** of inverse semigroups and morphisms is isomorphic to the category **G** of inductive groupoids and inductive functors.
◀

Green's star equivalences

Definition

- Given a semigroup S, for any $a, b \in S$,
  $a\,\mathcal{L}^*\,b$ if $a\,\mathcal{L}\,b$ in a semigroup T such that $S \subseteq T$.
  Dually, The relation $\mathcal{R}^*$on S is defined.
  $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$.
  $\mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^*$.



$\mathcal{L} \subseteq \mathcal{L}^*$

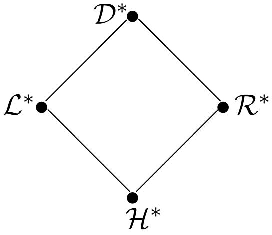
$\mathcal{R} \subseteq \mathcal{R}^*$

- A semigroup is called U-abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent.

Green's star equivalences

Definition

- Given a semigroup S, for any $a, b \in S$,
  $a \mathcal{L}^* b$ if $a \mathcal{L} b$ in a semigroup T such that $S \subseteq T$.
  Dually, The relation $\mathcal{R}^*$on S is defined.
  $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$.
  $\mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^*$.



$\mathcal{L} \subseteq \mathcal{L}^*$

$\mathcal{R} \subseteq \mathcal{R}^*$

- A semigroup is called *U-abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent.

(IC) and concordant semigroups

Let S be an abundant semigroup with $E(S) = U$. If $a \in S$, then a^* and $a^\dagger$ denote typical idempotents in L_a^* and R_a^*, respectively.

Then S is idempotent-connected (IC) if for any $a \in S$ and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger \rangle$, where for any $e \in U$, $\langle e \rangle$ is the subsemigroup generated by idempotents of eUe.

An abundant semigroup S is called a **concordant semigroup** if S satisfies Condition (IC) and its set of idempotents generates a regular subsemigroup.

◀

(IC) and concordant semigroups

Let S be an abundant semigroup with $E(S) = U$. If $a \in S$, then a^* and $a^\dagger$ denote typical idempotents in L_a^* and R_a^*, respectively.

Then S is idempotent-connected (IC) if for any $a \in S$ and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger \rangle$, where for any $e \in U$, $\langle e \rangle$ is the subsemigroup generated by idempotents of eUe.

An abundant semigroup S is called a **concordant semigroup** if S satisfies Condition (IC) and its set of idempotents generates a regular subsemigroup.

(IC) and concordant semigroups

Let S be an abundant semigroup with $E(S) = U$. If $a \in S$, then a^* and $a^\dagger$ denote typical idempotents in L_a^* and R_a^*, respectively.

Then S is idempotent-connected (IC) if for any $a \in S$ and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger \rangle$, where for any $e \in U$, $\langle e \rangle$ is the subsemigroup generated by idempotents of eUe.

An abundant semigroup S is called a **concordant semigroup** if S satisfies Condition (IC) and its set of idempotents generates a regular subsemigroup.

(IC) and concordant semigroups

Let S be an abundant semigroup with $E(S) = U$. If $a \in S$, then a^* and $a^\dagger$ denote typical idempotents in L^*_a and R^*_a, respectively.

Then S is idempotent-connected (IC) if for any $a \in S$ and for some $a^\dagger$, a^*, there exists a bijection $\alpha : \langle a^\dagger \rangle \to \langle a^* \rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^\dagger \rangle$, where for any $e \in U$, $\langle e \rangle$ is the subsemigroup generated by idempotents of eUe.

An abundant semigroup S is called a **concordant semigroup** if S satisfies Condition (IC) and its set of idempotents generates a regular subsemigroup.

◀

Concordant semigroups $\to$ Categories

Let S be a concordant semigroup with set of idempotents U.
Set

$$\mathbf{K}(S) = \{(e, a, f) : f \in L_a^* \cap U, e \in R_a^* \cap U\}.$$

We define a partial binary operation on $K(S)$ by

$$(e, a, f) \cdot (g, b, h) = \begin{cases} (e, ab, h) & \text{if } f = g \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Partial Order:
$(e, a, f) \leq (g, b, h) \Leftrightarrow e \; \omega \; g, a = eb$ and $f = e\beta$, where $\beta : \langle e \rangle \to \langle f \rangle$.

Fact 4: The set $(\mathbf{K}(S), \cdot, \leq)$ forms an inductive cancellative category.

Concordant semigroups → Categories

Let S be a concordant semigroup with set of idempotents U.
Set

$$\mathbf{K}(S) = \{(e, a, f) : f \in L_a^* \cap U, e \in R_a^* \cap U\}.$$

We define a partial binary operation on $K(S)$ by

$$(e, a, f) \cdot (g, b, h) = \begin{cases} (e, ab, h) & \text{if } f = g \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Partial Order:
$(e, a, f) \leq (g, b, h) \Leftrightarrow e \ \omega \ g, a = eb$ and $f = e\beta$, where $\beta : \langle e \rangle \to \langle f \rangle$.

Fact 4: The set $(\mathbf{K}(S), \cdot, \leq)$ forms an inductive cancellative category.

Concordant semigroups $\rightarrow$ Categories

Let S be a concordant semigroup with set of idempotents U.
Set

$$\mathbf{K}(S) = \{(e, a, f) : f \in L_a^* \cap U, e \in R_a^* \cap U\}.$$

We define a partial binary operation on $K(S)$ by

$$(e, a, f) \cdot (g, b, h) = \begin{cases} (e, ab, h) & \text{if } f = g \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Partial Order:
$(e, a, f) \leq (g, b, h) \Leftrightarrow e \ \omega \ g, a = eb$ and $f = e\beta$, where $\beta : \langle e \rangle \rightarrow \langle f \rangle$.

Fact 4: The set $(\mathbf{K}(S), \cdot, \leq)$ forms an inductive cancellative category.

Categories → Concordant semigroups

Let $(C, \cdot, \leq)$ be an inductive cancellative category with regular biordered set of objects E and ρ be a congruence on $(C, \cdot, \leq)$.

For any $x \in C$, $\bar{x}$ denotes the ρ-class containing x. Define

$$\bar{x} \circ \bar{y} = \overline{(x \otimes y)_h},$$

where $h \in S(\mathbf{r}(x), \mathbf{d}(y))$.

Fact 5: The set $(C/\rho, \circ)$ forms a concordant semigroup.

Theorem: The category of inductive cancellative categories is equivalent to the category of concordant semigroups. (S. Armstrong, 1988)

◀

Categories → Concordant semigroups

Let $(C, \cdot, \leq)$ be an inductive cancellative category with regular biordered set of objects E and ρ be a congruence on $(C, \cdot, \leq)$.

For any $x \in C$, $\bar{x}$ denotes the ρ-class containing x. Define

$$\bar{x} \circ \bar{y} = \overline{(x \otimes y)_h},$$

where $h \in S(\mathbf{r}(x), \mathbf{d}(y))$.

Fact 5: The set $(C/\rho, \circ)$ forms a concordant semigroup.

Theorem: The category of inductive cancellative categories is equivalent to the category of concordant semigroups. (S. Armstrong, 1988)

Let $(C, \cdot, \leq)$ be an inductive cancellative category with regular biordered set of objects E and ρ be a congruence on $(C, \cdot, \leq)$.

For any $x \in C$, $\bar{x}$ denotes the ρ-class containing x. Define

$$\bar{x} \circ \bar{y} = \overline{(x \otimes y)_h},$$

where $h \in S(\mathbf{r}(x), \mathbf{d}(y))$.

Fact 5: The set $(C/\rho, \circ)$ forms a concordant semigroup.

Theorem: The category of inductive cancellative categories is equivalent to the category of concordant semigroups. (S. Armstrong, 1988)

◀

$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \, \widetilde{\mathcal{L}}_U \, b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
$$a \, \widetilde{\mathcal{R}}_U \, b \Leftrightarrow (\forall e \in U)(ea = a \text{ if and only if } eb = b),$$
$$\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{L}}_U \wedge \widetilde{\mathcal{R}}_U, \qquad \widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \vee \widetilde{\mathcal{R}}_U.$$

- A semigroup S with $U \subseteq E(S)$ is said to be weakly U-abundant if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies **Congruence Condition** (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.
- A weakly E-abundant semigroup (S) is an **Ehresmann semigroup** if S satisfies (C) and E is a semilattice.
- A **weakly U-concordant semigroup** is a weakly U-abundant semigroup with (C) and $\langle U \rangle$ being a regular semigroup whose set of idempotents is U.

$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a\,\widetilde{\mathcal{L}}_U\,b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
$$a\,\widetilde{\mathcal{R}}_U\,b \Leftrightarrow (\forall e \in U)(ea = a \text{ if and only if } eb = b),$$
$$\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{L}}_U \wedge \widetilde{\mathcal{R}}_U, \qquad \widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \vee \widetilde{\mathcal{R}}_U.$$

- A semigroup S with $U \subseteq E(S)$ is said to be weakly U-abundant if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies **Congruence Condition** (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.
- A weakly E-abundant semigroup (S) is an **Ehresmann semigroup** if S satisfies (C) and E is a semilattice.
- A **weakly U-concordant semigroup** is a weakly U-abundant semigroup with (C) and $\langle U \rangle$ being a regular semigroup whose set of idempotents is U.

$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \, \widetilde{\mathcal{L}}_U \, b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
$$a \, \widetilde{\mathcal{R}}_U \, b \Leftrightarrow (\forall e \in U)(ea = a \text{ if and only if } eb = b),$$
$$\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{L}}_U \wedge \widetilde{\mathcal{R}}_U, \qquad \widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \vee \widetilde{\mathcal{R}}_U.$$

- A semigroup S with $U \subseteq E(S)$ is said to be weakly U-abundant if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies **Congruence Condition** (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.
- A weakly E-abundant semigroup (S) is an **Ehresmann semigroup** if S satisfies (C) and E is a semilattice.
- A **weakly U-concordant semigroup** is a weakly U-abundant semigroup with (C) and $\langle U \rangle$ being a regular semigroup whose set of idempotents is U.

$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Let $U \subseteq E(S)$. For any $a, b \in S$,
$$a\,\widetilde{\mathcal{L}}_U\,b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
$$a\,\widetilde{\mathcal{R}}_U\,b \Leftrightarrow (\forall e \in U)(ea = a \text{ if and only if } eb = b),$$
$$\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{L}}_U \wedge \widetilde{\mathcal{R}}_U, \qquad \widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \vee \widetilde{\mathcal{R}}_U.$$
- A semigroup S with $U \subseteq E(S)$ is said to be weakly U-abundant if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies **Congruence Condition** (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.
- A weakly E-abundant semigroup (S) is an **Ehresmann semigroup** if S satisfies (C) and E is a semilattice.
- A **weakly U-concordant semigroup** is a weakly U-abundant semigroup with (C) and $\langle U \rangle$ being a regular semigroup whose set of idempotents is U.

$\widetilde{\mathcal{L}}_U$, $\widetilde{\mathcal{R}}_U$ equivalences

Definition

- Let $U \subseteq E(S)$. For any $a, b \in S$,
$$a \,\widetilde{\mathcal{L}}_U\, b \Leftrightarrow (\forall e \in U)(ae = a \text{ if and only if } be = b),$$
$$a \,\widetilde{\mathcal{R}}_U\, b \Leftrightarrow (\forall e \in U)(ea = a \text{ if and only if } eb = b),$$
$$\widetilde{\mathcal{H}}_U = \widetilde{\mathcal{L}}_U \wedge \widetilde{\mathcal{R}}_U, \qquad \widetilde{\mathcal{D}}_U = \widetilde{\mathcal{L}}_U \vee \widetilde{\mathcal{R}}_U.$$
- A semigroup S with $U \subseteq E(S)$ is said to be weakly U-abundant if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U.
- A weakly U-abundant semigroup S satisfies **Congruence Condition** (C) if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.
- A weakly E-abundant semigroup (S) is an **Ehresmann semigroup** if S satisfies (C) and E is a semilattice.
- A **weakly U-concordant semigroup** is a weakly U-abundant semigroup with (C) and $\langle U \rangle$ being a regular semigroup whose set of idempotents is U.

Ehresmann semigroups → Categories

Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E.

$$C(S) = (S, \cdot)$$

where,$\cdot$ is defined by the rule that for any $x, y \in S$,

$$x \cdot y = \begin{cases} xy & \text{if } \; x^* = y^\dagger \\ \text{undefined} & \text{otherwise.} \end{cases}$$

where xy is the product of x and y in S.

Partial orders $x \leq_r y \Leftrightarrow x = x^\dagger y$; $\quad x \leq_l y \Leftrightarrow x = yx^*$

Fact 6: The set $(C(S), \cdot, \leq_r, \leq_l)$ forms an Ehresmann category.

Ehresmann semigroups → Categories

Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E.

$$C(S) = (S, \cdot)$$

where,$\cdot$ is defined by the rule that for any $x, y \in S$,

$$x \cdot y = \begin{cases} xy & \text{if } \; x^* = y^\dagger \\ \text{undefined} & \text{otherwise.} \end{cases}$$

where xy is the product of x and y in S.

Partial orders $x \leq_r y \Leftrightarrow x = x^\dagger y$; $\quad x \leq_l y \Leftrightarrow x = yx^*$

Fact 6: The set $(C(S), \cdot, \leq_r, \leq_l)$ forms an Ehresmann category.

Let S be an Ehresmann semigroup with distinguished semilattice of idempotents E.

$$C(S) = (S, \cdot)$$

where,$\cdot$ is defined by the rule that for any $x, y \in S$,

$$x \cdot y = \begin{cases} xy & \text{if } \; x^* = y^\dagger \\ \text{undefined} & \text{otherwise.} \end{cases}$$

where xy is the product of x and y in S.

Partial orders $x \leq_r y \Leftrightarrow x = x^\dagger y$; $\quad x \leq_l y \Leftrightarrow x = yx^*$

Fact 6: The set $(C(S), \cdot, \leq_r, \leq_l)$ forms an Ehresmann category.

Let $(C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category. We define a *pseudo-product* $\otimes$ on C by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Fact 7: The set $(C, \otimes)$ is an Ehresmann semigroup.

Theorem: The category of Ehresmann semigroups and admissible homomorphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors. (M.V.Lawson, 1989)

◀

Let $(C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category. We define a *pseudo-product* $\otimes$ on C by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Fact 7: The set $(C, \otimes)$ is an Ehresmann semigroup.

Theorem: The category of Ehresmann semigroups and admissible homomorphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors. (M.V.Lawson, 1989)

◀

Let $(C, \cdot, \leq_r, \leq_l)$ be an Ehresmann category. We define a *pseudo-product* $\otimes$ on C by the rule that

$$a \otimes b = (a|\mathbf{r}(a) \wedge \mathbf{d}(b)) \cdot (\mathbf{r}(a) \wedge \mathbf{d}(b)|b).$$

Fact 7: The set $(C, \otimes)$ is an Ehresmann semigroup.

Theorem: The category of Ehresmann semigroups and admissible homomorphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors. (M.V.Lawson, 1989)

◀

Proper categories

Let C be a category over a regular biordered set E. Choose only one element from each $\mathcal{L}$-class and each $\mathcal{R}$-class of E containing e, denoted by $e^{\star}$ and e^{+}, respectively. Suppose also

1. if $e \,\mathcal{R}\, f$ or $e \,\mathcal{L}\, f$, then $\exists[e,f]$ in $\mathrm{Hom}(e,f)$ s.t. $[e,e] = 1_e$;
2. if $x \in \mathcal{C}$, $h \in E$ and $h \,\omega^l\, \mathbf{d}_x$, then there exists a **restriction** $h|x$ such that $\mathbf{d}_{h|x} = h$ and $\mathbf{r}_{h|x} \,\omega^l\, \mathbf{r}_x$, in particular, $\mathbf{r}_{\mathbf{d}_x|x} \,\mathcal{L}\, \mathbf{r}_x$ and $x \cdot [\mathbf{r}_x, \mathbf{r}_{\mathbf{d}_x|x}] = \mathbf{d}_x|x$;
3. if $e \mathcal{R} f \;\mathcal{R}\; g$ or $e \mathcal{L} f \;\mathcal{L}\; g$, then $[e,f] \cdot [f,g] = [e,g]$;
4. if $g \;\omega\; e$ and $e \mathcal{R} f$ or $e \mathcal{L} f$, then $g|[e,f] = [g, gf] \cdot [gf, (gf)^{\star}]$;
5. if $e \,\omega^l\, f \,\omega^l\, \mathbf{d}_x$, then $e|(f|x) = e|x$;
6. if $h \,\omega^l\, \mathbf{d}_x$ and $\exists x \cdot y$, then $h|(x \cdot y) = (h|x) \cdot (g|y)$, where $g = \mathbf{r}_{h|x}$.

Then C is a **proper category**.

Proper categories

Let C be a category over a regular biordered set E. Choose only one element from each $\mathcal{L}$-class and each $\mathcal{R}$-class of E containing e, denoted by $e^{\star}$ and e^{+}, respectively. Suppose also

1. if $e \mathcal{R} f$ or $e \mathcal{L} f$, then $\exists [e,f]$ in $\text{Hom}(e,f)$ s.t. $[e,e] = 1_e$;
2. if $x \in \mathcal{C}$, $h \in E$ and $h \, \omega^l \, \mathbf{d}_x$, then there exists a **restriction** $h|x$ such that $\mathbf{d}_{h|x} = h$ and $\mathbf{r}_{h|x} \, \omega^l \, \mathbf{r}_x$, in particular, $\mathbf{r}_{\mathbf{d}_x|x} \, \mathcal{L} \, \mathbf{r}_x$ and $x \cdot [\mathbf{r}_x, \mathbf{r}_{\mathbf{d}_x|x}] = \mathbf{d}_x|x$;
3. if $e \mathcal{R} f \, \mathcal{R} \, g$ or $e \mathcal{L} f \, \mathcal{L} \, g$, then $[e,f] \cdot [f,g] = [e,g]$;
4. if $g \, \omega \, e$ and $e \mathcal{R} f$ or $e \mathcal{L} f$, then $g|[e,f] = [g,gf] \cdot [gf,(gf)^{\star}]$;
5. if $e \, \omega^l \, f \, \omega^l \, \mathbf{d}_x$, then $e|(f|x) = e|x$;
6. if $h \, \omega^l \, \mathbf{d}_x$ and $\exists x \cdot y$, then $h|(x \cdot y) = (h|x) \cdot (g|y)$, where $g = \mathbf{r}_{h|x}$.

Then C is a **proper category**.

Proper categories

Let C be a category over a regular biordered set E. Choose only one element from each $\mathcal{L}$-class and each $\mathcal{R}$-class of E containing e, denoted by $e^\star$ and e^+, respectively. Suppose also

1. if $e \mathcal{R} f$ or $e \mathcal{L} f$, then $\exists[e, f]$ in $\mathrm{Hom}(e, f)$ s.t. $[e, e] = 1_e$;
2. if $x \in \mathcal{C}$, $h \in E$ and $h \, \omega^l \, \mathbf{d}_x$, then there exists a **restriction** $h|x$ such that $\mathbf{d}_{h|x} = h$ and $\mathbf{r}_{h|x} \, \omega^l \, \mathbf{r}_x$, in particular, $\mathbf{r}_{\mathbf{d}_x|x} \, \mathcal{L} \, \mathbf{r}_x$ and $x \cdot [\mathbf{r}_x, \mathbf{r}_{\mathbf{d}_x|x}] = \mathbf{d}_x|x$;
3. if $e \mathcal{R} f \, \mathcal{R} \, g$ or $e \mathcal{L} f \, \mathcal{L} \, g$, then $[e, f] \cdot [f, g] = [e, g]$;
4. if $g \, \omega \, e$ and $e \mathcal{R} f$ or $e \mathcal{L} f$, then $g|[e, f] = [g, gf] \cdot [gf, (gf)^\star]$;
5. if $e \, \omega^l \, f \, \omega^l \, \mathbf{d}_x$, then $e|(f|x) = e|x$;
6. if $h \, \omega^l \, \mathbf{d}_x$ and $\exists x \cdot y$, then $h|(x \cdot y) = (h|x) \cdot (g|y)$, where $g = \mathbf{r}_{h|x}$.

Then C is a **proper category**.

Proper categories

Let C be a category over a regular biordered set E. Choose only one element from each $\mathcal{L}$-class and each $\mathcal{R}$-class of E containing e, denoted by $e^\star$ and e^+, respectively. Suppose also

1. if $e \,\mathcal{R}\, f$ or $e \,\mathcal{L}\, f$, then $\exists [e,f]$ in $\mathrm{Hom}(e,f)$ s.t. $[e,e] = 1_e$;
2. if $x \in \mathcal{C}$, $h \in E$ and $h \,\omega^l\, \mathbf{d}_x$, then there exists a **restriction** $h|x$ such that $\mathbf{d}_{h|x} = h$ and $\mathbf{r}_{h|x} \,\omega^l\, \mathbf{r}_x$, in particular, $\mathbf{r}_{\mathbf{d}_x|x} \,\mathcal{L}\, \mathbf{r}_x$ and $x \cdot [\mathbf{r}_x, \mathbf{r}_{\mathbf{d}_x|x}] = \mathbf{d}_x|x$;
3. if $e \mathcal{R} f \,\mathcal{R}\, g$ or $e \mathcal{L} f \,\mathcal{L}\, g$, then $[e,f] \cdot [f,g] = [e,g]$;
4. if $g \,\omega\, e$ and $e \mathcal{R} f$ or $e \mathcal{L} f$, then $g|[e,f] = [g,gf] \cdot [gf,(gf)^\star]$;
5. if $e \,\omega^l\, f \,\omega^l\, \mathbf{d}_x$, then $e|(f|x) = e|x$;
6. if $h \,\omega^l\, \mathbf{d}_x$ and $\exists\, x \cdot y$, then $h|(x \cdot y) = (h|x) \cdot (g|y)$, where $g = \mathbf{r}_{h|x}$.

Then C is a **proper category**.

Proper categories

Let C be a category over a regular biordered set E. Choose only one element from each $\mathcal{L}$-class and each $\mathcal{R}$-class of E containing e, denoted by $e^\star$ and e^+, respectively. Suppose also

1. if $e\,\mathcal{R}\,f$ or $e\,\mathcal{L}\,f$, then $\exists[e,f]$ in $\mathrm{Hom}(e,f)$ s.t. $[e,e]=1_e$;
2. if $x\in\mathcal{C}$, $h\in E$ and $h\,\omega^l\,\mathbf{d}_x$, then there exists a **restriction** $h|x$ such that $\mathbf{d}_{h|x}=h$ and $\mathbf{r}_{h|x}\,\omega^l\,\mathbf{r}_x$, in particular, $\mathbf{r}_{\mathbf{d}_x|x}\,\mathcal{L}\,\mathbf{r}_x$ and $x\cdot[\mathbf{r}_x,\mathbf{r}_{\mathbf{d}_x|x}]=\mathbf{d}_x|x$;
3. if $e\,\mathcal{R}\,f\,\mathcal{R}\,g$ or $e\,\mathcal{L}\,f\,\mathcal{L}\,g$, then $[e,f]\cdot[f,g]=[e,g]$;
4. if $g\,\omega\,e$ and $e\,\mathcal{R}\,f$ or $e\,\mathcal{L}\,f$, then $g|[e,f]=[g,gf]\cdot[gf,(gf)^\star]$;
5. if $e\,\omega^l\,f\,\omega^l\,\mathbf{d}_x$, then $e|(f|x)=e|x$;
6. if $h\,\omega^l\,\mathbf{d}_x$ and $\exists\,x\cdot y$, then $h|(x\cdot y)=(h|x)\cdot(g|y)$, where $g=\mathbf{r}_{h|x}$.

Then C is a **proper category**.

Categories $\to$ Weakly U-concordant semigroups

Let $(P, \cdot)$ be a strongly proper category with $\mathrm{Ob}(P) = U$ a regular biordered set.

Suppose that $x \in P$, $h\ \omega^r\ \mathbf{d}_x$ and $k\ \omega^l\ \mathbf{r}_x$. We define that

$$h * x = [h, h\mathbf{d}_x] \cdot h\mathbf{d}_x|x, \text{ and } x * k = x|\mathbf{r}_x k \cdot [\mathbf{r}_x k, k].$$

Define a relation ρ on P by the rule that for any $x, y \in P$,

$$x\,\rho\,y \Leftrightarrow \mathbf{d}_x\,\mathcal{R}\,\mathbf{d}_y, \mathbf{r}_x\,\mathcal{L}\,\mathbf{r}_y \text{ and } x \cdot [\mathbf{r}_x, \mathbf{r}_y] = [\mathbf{d}_x, \mathbf{d}_y] \cdot y.$$

For $x, y \in P$, $h \in S(\mathbf{r}_x, \mathbf{d}_y)$, we define that

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denote the ρ-class of x in P and $(x \otimes y)_h = (x * h) \cdot (h * y)$.

Fact 8: The set $\mathcal{S}(P) = (P/\rho, \odot)$ forms a weakly U-concordant semigroup.

Categories $\to$ Weakly U-concordant semigroups

Let $(P, \cdot)$ be a strongly proper category with $\mathrm{Ob}(P) = U$ a regular biordered set.
Suppose that $x \in P$, $h \ \omega^r \ \mathbf{d}_x$ and $k \ \omega^l \ \mathbf{r}_x$. We define that

$$h * x = [h, h\mathbf{d}_x] \cdot h\mathbf{d}_x|x, \text{ and } x * k = x|\mathbf{r}_x k \cdot [\mathbf{r}_x k, k].$$

Define a relation ρ on P by the rule that for any $x, y \in P$,

$$x \,\rho\, y \Leftrightarrow \mathbf{d}_x \,\mathcal{R}\, \mathbf{d}_y, \mathbf{r}_x \,\mathcal{L}\, \mathbf{r}_y \text{ and } x \cdot [\mathbf{r}_x, \mathbf{r}_y] = [\mathbf{d}_x, \mathbf{d}_y] \cdot y.$$

For $x, y \in P$, $h \in S(\mathbf{r}_x, \mathbf{d}_y)$, we define that

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denote the ρ-class of x in P and $(x \otimes y)_h = (x * h) \cdot (h * y)$.

Fact 8: The set $\mathcal{S}(P) = (P/\rho, \odot)$ forms a weakly U-concordant semigroup.

Categories → Weakly U-concordant semigroups

Let $(P, \cdot)$ be a strongly proper category with $\mathrm{Ob}(P) = U$ a regular biordered set.
Suppose that $x \in P$, $h\ \omega^r\ \mathbf{d}_x$ and $k\ \omega^l\ \mathbf{r}_x$. We define that

$$h * x = [h, h\mathbf{d}_x] \cdot h\mathbf{d}_x|x, \text{ and } x * k = x|\mathbf{r}_x k \cdot [\mathbf{r}_x k, k].$$

Define a relation ρ on P by the rule that for any $x, y \in P$,

$$x\,\rho\,y \Leftrightarrow \mathbf{d}_x\,\mathcal{R}\,\mathbf{d}_y, \mathbf{r}_x\,\mathcal{L}\,\mathbf{r}_y \text{ and } x \cdot [\mathbf{r}_x, \mathbf{r}_y] = [\mathbf{d}_x, \mathbf{d}_y] \cdot y.$$

For $x, y \in P$, $h \in S(\mathbf{r}_x, \mathbf{d}_y)$, we define that

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denote the ρ-class of x in P and $(x \otimes y)_h = (x * h) \cdot (h * y)$.

Fact 8: The set $\mathcal{S}(P) = (P/\rho, \odot)$ forms a weakly U-concordant semigroup.

Let $(P, \cdot)$ be a strongly proper category with $\text{Ob}(P) = U$ a regular biordered set.
Suppose that $x \in P$, $h \ \omega^r \ \mathbf{d}_x$ and $k \ \omega^l \ \mathbf{r}_x$. We define that

$$h * x = [h, h\mathbf{d}_x] \cdot h\mathbf{d}_x|x, \text{ and } x * k = x|\mathbf{r}_x k \cdot [\mathbf{r}_x k, k].$$

Define a relation ρ on P by the rule that for any $x, y \in P$,

$$x \, \rho \, y \Leftrightarrow \mathbf{d}_x \, \mathcal{R} \, \mathbf{d}_y, \mathbf{r}_x \, \mathcal{L} \, \mathbf{r}_y \text{ and } x \cdot [\mathbf{r}_x, \mathbf{r}_y] = [\mathbf{d}_x, \mathbf{d}_y] \cdot y.$$

For $x, y \in P$, $h \in S(\mathbf{r}_x, \mathbf{d}_y)$, we define that

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denote the ρ-class of x in P and $(x \otimes y)_h = (x * h) \cdot (h * y)$.

Fact 8: The set $\mathcal{S}(P) = (P/\rho, \odot)$ forms a weakly U-concordant semigroup.

Let $(P, \cdot)$ be a strongly proper category with $\mathrm{Ob}(P) = U$ a regular biordered set.
Suppose that $x \in P$, $h \ \omega^r \ \mathbf{d}_x$ and $k \ \omega^l \ \mathbf{r}_x$. We define that

$$h * x = [h, h\mathbf{d}_x] \cdot h\mathbf{d}_x | x, \text{ and } x * k = x | \mathbf{r}_x k \cdot [\mathbf{r}_x k, k].$$

Define a relation ρ on P by the rule that for any $x, y \in P$,

$$x \, \rho \, y \Leftrightarrow \mathbf{d}_x \, \mathcal{R} \, \mathbf{d}_y, \mathbf{r}_x \, \mathcal{L} \, \mathbf{r}_y \text{ and } x \cdot [\mathbf{r}_x, \mathbf{r}_y] = [\mathbf{d}_x, \mathbf{d}_y] \cdot y.$$

For $x, y \in P$, $h \in S(\mathbf{r}_x, \mathbf{d}_y)$, we define that

$$\bar{x} \odot \bar{y} = \overline{(x \otimes y)_h},$$

where $\bar{x}$ denote the ρ-class of x in P and $(x \otimes y)_h = (x * h) \cdot (h * y)$.

Fact 8: The set $\mathcal{S}(P) = (P/\rho, \odot)$ forms a weakly U-concordant semigroup.

Weakly U-concordant semigroups $\rightarrow$ Categories

Let S be a weakly U-concordant semigroup. Set

$$\mathbf{C}(S) = \{(e,x,f) : e \, \widetilde{\mathcal{R}}_U \, x \, \widetilde{\mathcal{L}}_U \, f, \quad e,f \in U\},$$

and define a partial binary operation by the rule that

$$(e,x,f)\cdot(u,y,v) = \begin{cases} (e,xy,v) & \text{if } f = u \\ \text{undefine} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S.

If $e \; \mathcal{R} \; f$ or $e \; \mathcal{L} \; f$, then $[e,f] = (e,ef,f)$.

Pre-orders: For any $(e,x,f) \in C(S)$ and $u,v \in U$ with $u \leq_l e$ and $v \leq_r f$, we define that

$$u|(e,x,f) = (u,ux,(ux)^{\star}) \text{ and } (e,x,f)|v = ((xv)^{+},xv,v).$$

Fact 9: The set $(\mathbf{C}(S),\cdot)$ forms a strongly proper category with restrictions and co-restrictions as above.

Weakly U-concordant semigroups $\rightarrow$ Categories

Let S be a weakly U-concordant semigroup. Set

$$\mathbf{C}(S) = \{(e, x, f) : e \, \widetilde{\mathcal{R}}_U \, x \, \widetilde{\mathcal{L}}_U \, f, \quad e, f \in U\},$$

and define a partial binary operation by the rule that

$$(e, x, f) \cdot (u, y, v) = \begin{cases} (e, xy, v) & \text{if } f = u \\ \text{undefine} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S.

If $e \, \mathcal{R} \, f$ or $e \, \mathcal{L} \, f$, then $[e, f] = (e, ef, f)$.

Pre-orders: For any $(e, x, f) \in C(S)$ and $u, v \in U$ with $u \leq_l e$ and $v \leq_r f$, we define that

$$u|(e, x, f) = (u, ux, (ux)^{\star}) \text{ and } (e, x, f)|v = ((xv)^{+}, xv, v).$$

Fact 9: The set $(\mathbf{C}(S), \cdot)$ forms a strongly proper category with restrictions and co-restrictions as above.

Let S be a weakly U-concordant semigroup. Set

$$\mathbf{C}(S) = \{(e, x, f) : e \ \widetilde{\mathcal{R}}_U \ x \ \widetilde{\mathcal{L}}_U \ f, \quad e, f \in U\},$$

and define a partial binary operation by the rule that

$$(e, x, f) \cdot (u, y, v) = \begin{cases} (e, xy, v) & \text{if } f = u \\ \text{undefine} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S.

If $e \ \mathcal{R} \ f$ or $e \ \mathcal{L} \ f$, then $[e, f] = (e, ef, f)$.

Pre-orders: For any $(e, x, f) \in C(S)$ and $u, v \in U$ with $u \leq_l e$ and $v \leq_r f$, we define that

$$u|(e, x, f) = (u, ux, (ux)^{\star}) \text{ and } (e, x, f)|v = ((xv)^{+}, xv, v).$$

Fact 9: The set $(\mathbf{C}(S), \cdot)$ forms a strongly proper category with restrictions and co-restrictions as above.

Weakly U-concordant semigroups $\rightarrow$ Categories

Let S be a weakly U-concordant semigroup. Set

$$\mathbf{C}(S) = \{(e, x, f) : e \ \widetilde{\mathcal{R}}_U \ x \ \widetilde{\mathcal{L}}_U \ f, \quad e, f \in U\},$$

and define a partial binary operation by the rule that

$$(e, x, f) \cdot (u, y, v) = \begin{cases} (e, xy, v) & \text{if } f = u \\ \text{undefine} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S.

If $e \ \mathcal{R} \ f$ or $e \ \mathcal{L} \ f$, then $[e, f] = (e, ef, f)$.

Pre-orders: For any $(e, x, f) \in C(S)$ and $u, v \in U$ with $u \leq_l e$ and $v \leq_r f$, we define that

$$u|(e, x, f) = (u, ux, (ux)^{\star}) \text{ and } (e, x, f)|v = ((xv)^{+}, xv, v).$$

Fact 9: The set $(\mathbf{C}(S), \cdot)$ forms a strongly proper category with restrictions and co-restrictions as above.

Weakly U-concordant semigroups $\to$ Categories

Let S be a weakly U-concordant semigroup. Set

$$\mathbf{C}(S) = \{(e, x, f) : e \; \widetilde{\mathcal{R}}_U \; x \; \widetilde{\mathcal{L}}_U \; f, \quad e, f \in U\},$$

and define a partial binary operation by the rule that

$$(e, x, f) \cdot (u, y, v) = \begin{cases} (e, xy, v) & \text{if } f = u \\ \text{undefine} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in S.

If $e \; \mathcal{R} \; f$ or $e \; \mathcal{L} \; f$, then $[e, f] = (e, ef, f)$.

Pre-orders: For any $(e, x, f) \in C(S)$ and $u, v \in U$ with $u \leq_l \; e$ and $v \; \leq_r f$, we define that

$$u|(e, x, f) = (u, ux, (ux)^{\star}) \text{ and } (e, x, f)|v = ((xv)^{+}, xv, v).$$

Fact 9: The set $(\mathbf{C}(S), \cdot)$ forms a strongly proper category with restrictions and co-restrictions as above.

Correspondence

Theorem The category of weakly U-concordant semigroups and admissible morphisms is equivalent to the category of strongly proper categories and proper functors.

Theorem: The category of inductive cancellative categories is equivalent to the category of concordant semigroups. (S. Armstrong, 1988)

◀

Correspondence

Theorem The category of weakly U-concordant semigroups and admissible morphisms is equivalent to the category of strongly proper categories and proper functors.

Theorem: The category of inductive cancellative categories is equivalent to the category of concordant semigroups. (S. Armstrong, 1988)

◀