Degree 2 Transformation Semigroups as Continuous Maps on Graphs

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ENSBAN July 1, 2020

Introduction

In combinatorics and algebra morphisms are usually defined by preserving direct image:

"A morphism between graphs is a function between vertices that sends edges to edges."

"A simplicial map between simplicial complexes is a function between vertices that sends faces to faces."

But in topology, morphisms are defined by preservation of inverse image.

The purpose of the talk is to show how preservation of structures under inverse image arises naturally in semigroup theory in many ways.

We concentrate on finite graphs. As far as we know, this straightforward idea has not been explored elsewhere.

Degree of functions and transformation semigroups

Let $f \in PF_R(V)$, a partial function $f : V \to V$.

A $fiber$ of f is a subset F of V of the form $F = vf^{-1}$ for some $v \in V$.

The $degree$ of f, $\deg(f) = \max\{\mathrm{card}(F) | F \text{ a fiber of } f\}$.

A transformation semigroup (ts) is a pair (V, S) where S is a subsemigroup of $PF_R(V)$.

The $degree$ of (V, S), $\deg(V, S) = \max\{\deg(f) | f \in S\}$.

Basic Facts About Degrees

Let $f, g \in PF_R(V)$. Then $\text{deg}(fg) \leq \text{deg}(f)\text{deg}(g)$.

$v \overset{\times\text{deg}(g)}{\rightarrow} vg^{-1} \overset{\times\text{deg}(f)}{\rightarrow} (vg^{-1})f^{-1} = v(fg)^{-1}$.

It is easy to see that if $|V| > 2$, then $\{f \in PF_R(V) | \text{deg}(f) \leq k\}$ is a subsemigroup of $PF_R(V)$ iff $k \leq 1$.

In fact, if $V = \{1, \ldots n\}$, then $PF_R(V)$ is generated by its elements of degree at most 1, that is, the symmetric inverse monoid, and the idempotent of degree 2:

$$1, 2 \mapsto 1, 3 \mapsto 3, \ldots n \mapsto n.$$

Thus we need some combinatorial constraint to build ts of degree 2.

Continuous Functions on a Graph

Let $\Gamma = (V, E)$ be a (simple) graph. We write uv if $\{u, v\} \in E$.

$f \in PF_R(V)$ is *continuous* on Γ if $\forall x \in V \cup E$ then xf^{-1} is empty or $xf^{-1} \in V \cup E$.

We are treating Γ as a simplicial complex and continuous means that the inverse image of any face is a face.

Let $M(\Gamma)$ be the monoid of continuous functions on Γ. We have the ts $(V, M(\Gamma))$.

Theorem
$(V, M(\Gamma))$ *is a ts of degree at most 2.*

The Graph of Fibers of a ts

Let $X = (V, S)$ be a ts of degree at most 2.

The graph of fibers, $\Gamma(X) = (V, E)$ is the simple graph whose edges are the fibers of cardinality 2:

$$E = \{\{u, v\} | u \neq v, \exists f \in S, uf = vf\}$$

We have the following "Cayley" or "Preston-Wagner" Theorem for ts of degree at most 2:

Theorem

Let $X = (V, S)$ be a ts of degree at most 2. Then S is a subsemigroup of $M(\Gamma(X))$ and X embeds into $(V, M(\Gamma(X)))$.

The Structure of Continuous Functions on Graphs

Let $\Gamma = (V, E)$ be a graph. We note that $M(\Gamma)$ is closed under restriction of partial functions.

Indeed for all $X \subseteq V$, the partial identity $1|_X \in M(\Gamma)$. Thus for all $f \in M(\Gamma)$ both the restriction to the domain $1_X f$ and co-restriction to the range $f 1_X$ belong to $M(\Gamma)$.

Let $f \in M(\Gamma)$. Define $\mathcal{M}(f) = \{v \in V | \exists w \neq v, vf = wf\}$ be the union of the set of fibers of size 2 of f.

Define $Sing(f)$, the singular part of f, to be the restriction of the domain of f to $\mathcal{M}(f)$.

The Structure of Singular Functions on Graphs

Recall that a $matching$ in a graph $\Gamma = (V, E)$ is a set $M \subseteq E$ such that no two distinct edges in M have a vertex in common. An $anti - clique$ (or independent set) is a subset $A \subseteq V$ such that there are no edges between elements of A.

Theorem

Let $\Gamma = (V, E)$ be a graph and let $f \in M(\Gamma)$. Then the following are true.

(i) *$\mathcal{M}(f)$ is a matching of Γ.*

(ii) *$Im(Sing(f))$ is an anti-clique in Γ.*

(iii) *Conversely if $\mathcal{M} = \{e_1, \ldots, e_m\}$ is a matching of size m and $A = \{v_1, \ldots, v_m\}$ is an anti-clique of the same size m, then the function $f : V \to V$ with domain $\mathcal{M}$ and defined by $e_i f = v_i, i = 1, \ldots m$ is continuous and $f = Sing(f)$.*

The Structure of Injective Functions on Graphs

Let $\Gamma = (V, E)$ be a graph and $f \in M(\Gamma)$. The injective part of f, $Inj(f)$ is defined by $Inj(f) = f|_{Dom(f)-\mathcal{M}(f)}$.

A $graph\ morphism$ between graphs (V, E) and (V', E') is a function $f : V \to V'$ such that if $uv \in E$, then $(uf)(vf) \in E'$.

Note that a bijective graph morphism need not be an isomorphism of graphs, since not every edge in E' may be the image of an edge in E.

Let $X \subseteq V$. The induced graph $\Gamma(X)$ on X has X as vertices and all edges of Γ both of whose vertices are in X.

The Structure of Injective Functions on Graphs

Theorem

Let $\Gamma = (V, E)$ be a graph and $f \in M(\Gamma)$. Then the following are true.

(i) $Inj(f)$ *is a partial bijection.*

(ii) $Inj(f)^{-1}$ *is a bijective graph morphism from the induced graph on* $Image(Inj(f))$ *to the induced graph on* $Dom(Inj(f))$.

(iii) $Image(Inj(f)) \cap Image(Sing(f)) = \emptyset$ *and there are no edges between* $Image(Inj(f))$ *and* $Image(Sing(f))$. *(*$Proof$*: if* $u \in Image(Inj(f)), v \in Image(Sing(f))$ *and* $uv \in E$, *then Card*$(uv)f^{-1} = 3$, *contradicting continuity.)*

(iv) *Conversely, if* $X, Y \subseteq V$ *and* $g : \Gamma(X) \to \Gamma(Y)$ *is a bijective graph morphism, then* $f = g^{-1}$ *is continuous and* $f = Inj(f)$.

The Structure of Continuous Functions on a Graph

If f and g are partial functions on the same set with disjoint domains, then their join $f \vee g$ is the partial function whose domain is the union of the domains of f and g and such that $x(f \vee g) = xf$ if $x \in Dom(f)$ and $x(f \vee g) = xg$ if $x \in Dom(g)$.

In particular, $f = Sing(f) \vee Inj(f)$ is the join decomposition of f into its singular and injective parts.

The next Theorem characterizes which functions can serve as $Sing(f)$ and $Inj(f)$ for a continuous function on a graph.

The Structure of Continuous Functions on a Graph

Theorem

(i) *Let f be a continuous partial function on a graph Γ. Then the partition $ker(Sing(f))$ is a matching of Γ and the image $Im(Sing(f))$ is an anti-clique in Γ. Moreover, $Im(Sing(f)) \cap Im(Inj(f)) = \emptyset$, the inverse partial function of $Inj(f)$ is a bijective graph morphism onto its image, and there are no edges between a vertex in $Im(Sing(f))$ and a vertex in $Im(Inj(f))$.*

(ii) *Conversely, let g be a partial function such that $ker(g)$ is a matching in Γ and $Im(g)$ is an anti-clique in Γ. Let h be a partial bijection such that h^{-1} is a bijective graph morphism onto its image. Assume further that $Dom(g) \cap Dom(h) = Im(g) \cap Im(h) = \emptyset$ and that there are no edges between $Im(g)$ and $Im(h)$. Then $f = g \vee h$ is continuous and $g = Sing(f), h = Inj(f)$.*

Examples

*Example*1 **Empty Graphs**
Let V be a non-empty set and let $N(V)$ be the graph with no edges on V. Then a function f is continuous if and only if f is a partial bijection, that is, $|vf^{-1}| = 1$ for all $v \in Im(f)$. Therefore $M(N(V))$ is the symmetric inverse monoid on V consisting of all partial bijections on V.

$Example 2$ **Complete Graphs**

Let $n > 0$ and let K_n be the complete graph on n vertices $V_n = \{1, \ldots n\}$. Let f be a continuous function on K_n.

The image of $Sing(f)$ is an anti-clique and thus $|Im(Sing(f))| \leq 1$.

If $Im(Sing(f))$ is empty, then f is a partial bijection on V_n. Clearly any partial bijection is continuous on K_n.

Assume then that $Im(Sing(f)) = \{v\}$ for some $v \in V_n$.

We have seen that for all $w \in Im(Inj(f))$, vw is not an edge.

Therefore, $Inj(f)$ is the empty function and f is a partial constant function with domain an edge e.

Examples

Let $e \in E, v \in V$. Define $f_{e,v} : V \to V$ be the partial constant function with domain e that sends both vertices of e to v.

Then $f_{e,v}$ is continuous.

Therefore $M(K_n)$ consists of the symmetric inverse monoid on V together with the collection $\{f_{e,v} | e \in E, v \in V\}$.

Examples

Let $\Gamma = K_{n,n}$, the complete bipartite graph on $2n$ vertices, with bipartition $B_n \dot{\cup} W_n$ where $B_n = \{b_1 \ldots, b_n\}$ and $W_n = \{w_1, \ldots w_n\}$.

We compute the monoid of singular continuous functions $SM(K_{n,n})$, that is, those continuous functions of the form $f = Sing(f)$ of $K_{n,n}$.

A singular function on $K_{n,n}$ has a matching as its domain. It is

clear that a matching in $K_{n,n}$ can be identified with a partial bijection $M : B_n \to W_n$.

The corresponding matching is the set of all edges of the form $bM(b)$, where $b \in Dom(M)$.

Examples

The range of a singular function is an anti-clique and thus lies wholly in either B_n or W_n.

For each partial bijection M of rank k we have a unique singular function on $K_{n,n}$ by using the associated matching of M as fibers and sending them arbitrarily to either a k-set in B_n or a k-set in W_n.
This describes all singular continuous functions on $K_{n,n}$.

Decomposition and Complexity of degree 2 ts

The *complexity* of a finite semigroup S is the least number of non-trivial groups needed in order to represent S as a homomorphic image of a subsemigroup of a wreath product of groups and semigroups whose maximal subgroups are trivial.

Such a decomposition is guaranteed by the Krohn-Rhodes Theorem. The complexity problem is to compute this minimal number.

The complexity of $PF_R(V)$ is $|V| - 1$, so there are semigroups of each complexity $n \geq 0$.

Theorem

Let (V, S) be a ts of degree 1. Then S divides the wreath product $(\{0,1\}, \{0,1\}) \wr Sym(V)$. Consequently, the complexity of S is at most 1.

Decomposition and Complexity of degree 2 ts

Recall that a $right\ regular\ band$ is an idempotent semigroup S such that $xyx = yx$ for all $x, y \in S$. Equivalently S is a band in which $Green's$ $\mathcal{L}$-relation is trivial.

Theorem

Let (V, S) be a ts of degree 2. Then there is a semilattice T, a right regular band U and groups G_1, G_2 such that S divides $T \wr G_2 \wr U \wr G_1$. Consequently the complexity of S is at most 2.

The well known theorem of Frucht shows that every finite monoid is the endomorphism monoid of a finite graph in the category of graphs and graph morphisms. Since there are monoids of arbitrary complexity the collection of finite monoids that have faithful representations by ts of degree at most 2 is a proper collection of monoids.

More generally:

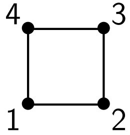
Theorem

Let (V, S) be a ts of degree k. Then there is a ts (Y, T) of degree $k-1$, a right regular band U and a group G such that S divides $T \wr U \wr G$. Consequently, by induction, the complexity of S is at most k.

It follows that for each $k > 0$ the collection of finite monoids that have faithful representations by ts of degree at most k is a proper collection of finite monoids.

The Smallest Semigroup of Complexity 2

We construct the smallest semigroup of complexity two as a semigroup of continuous maps on a 4-cycle:



There are two perfect matchings, namely, $12|34$ and $14|23$ and two maximal anti-cliques, 13 and 24.

All 8 possibilities of assigning a perfect matching to a maximal anti-clique defines a singular continuous map. Thus, for example, sending $12 \mapsto 3$, $34 \mapsto 1$ defines a continuous function and all 7 other possibilities do as well.

The Smallest Semigroup of Complexity 2

These 8 functions form a semigroup isomorphic to the simple semigroup $S = M(Z_2, \{1, 2\}, \{1, 2\}, C)$, where $C=$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

.

We add the identity and the permutation with cycle decomposition $z = (12)(34)$ which is an automorphism of Γ and thus an invertible continuous map. One can check that this is a 10 element monoid which is known to be the unique semigroup of order 10 of complexity 2 and that all semigroups of order at most 9, have complexity at most 1.

The "Link" to Translational Hulls of 0-Simple Semigroups

Let $S = \mathcal{M}^0(1, A, B, C)$ be a 0-simple finite semigroup over the trivial group. We write the matrix C in "inner product" notation: $C(b, a) =< b, a >$.

For our purposes, we define the translational hull $\Omega(S)$ to be the set of all ordered pairs (f, f^*), where $f \in PF_R(B)$, $f^* \in PF_L(A)$ which are "adjoint" with respect to C:

$$< bf, a >=< b, f^*a >, \forall a \in A, b \in B$$

.

These are the "linked equations."

The "Link" to Translational Hulls of 0-Simple Semigroups

Let $\Gamma = (V, E)$ be a graph. The graph incidence matrix of V is the $|V| \times |E|$ matrix $I = I(\Gamma)$ whose entry in position (v, e) is 1 if v is a vertex of e and 0 otherwise.

If we view Γ as a simplicial complex, we have the simplicial incidence matrix, which is the $|V| \times |E \cup V|$ matrix $C = C(\Gamma)$ with entries $C(v, w) = 1$ if and only if $v = w$ for $v, w \in V$ and as above, $C(v, e) = 1$ if and only if v is a vertex of the edge e. Thus, $C = [I|1_V]$, where 1_V is the $|V| \times |V|$ identity matrix. Let $S(\Gamma) = \mathcal{M}^0(1, E \cup V, V, C)$

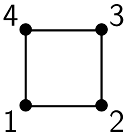
Theorem

Let $\Gamma = (V, E)$ be a graph. Then the translational hull of $S(\Gamma)$ is isomorphic to the monoid $M(\Gamma)$ of all continuous partial functions. on Γ.

Examples of Translational Hulls

Example 1 **Empty Graphs**
If $N(V)$ be the empty graph on V, then the matrix C is the identity matrix 1_V and thus $S(N(V))$ is the Brandt semigroup $M^0(1, V, V, 1_V)$.
We know that $M(N(V))$is the symmetric inverse semigroup on V and the previous result recovers the well-known result that the translational hull of the aperiodic Brandt semigroup on V is the symmetric inverse monoid on V.

Example 2 **The 4-cycle.**
Let Γ be the 4-cycle:



Then the matrix C of $S(\Gamma)$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

On the next slides we show the *Green* structure of $M(\Gamma)$.

Generalizations and Open Problems

The results of this paper can be generalized to ts of degree $k > 2$.

Let $\mathcal{S} = (V, H)$ be a simplicial complex of rank k, meaning that the number of vertices in its largest face is size k. A partial function $f : V \to V$ is continuous if the inverse image of each face of $\mathcal{S}$ is also a face.

Let $X = (V, S)$ be a ts of degree k.
A subset $W \subset V$ is *attached* if it is contained in some fiber of X.

Generalizations and Open Problems

Let H be the collection of attached subsets of V.
Then $\mathcal{S}(X) = (V, H)$ is a simplicial complex of rank k.
It is known that if $M(X)$ the monoid of continuous functions on a simplicial complex $\mathcal{S} = (V, H)$ of rank k, then $(V, M(X))$ is a ts of degree k.
Furthermore, if $X = (V, S)$ is a ts of degree k, then X embeds into the ts of continuous functions on the simplicial complex $\mathcal{S}(X)$.

As mentioned previously, the complexity of a degree k ts is at most k.

Generalizations and Open Problems

Problem 1 **Decidability of Complexity for ts of degree 2**
A ts of degree 2, that has a non-trivial subgroup has complexity 1 or 2.
Is there an algorithm to decide the complexity of a ts of degree 2?

In particular, is the lower bound in the paper, K. Henckell, J. Rhodes, and B. Steinberg. "An effective lower bound for the complexity of finite semigroups and automata" Trans. AMS, 364(4):18151857, 2012.

Generalizations and Open Problems

Problem 2 **Decidability of the variety and quasivariety of semigroups of degree 2**

A semigroup S has degree 2 if it has a faithful representation as a ts (Q, S) of degree 2.

The collection of all finite semigroups of degree 2 is closed under subsemigroups and direct product (by acting on the disjoint union.) Let Q_2 be the collection of finite semigroups of degree 2. Q_2 is a quasivariety of finite semigroups.

Is membership in Q_2 decidable? Does Q_2 have a finite basis of profinite implications?

Generalizations and Open Problems

Let V_2 be the variety of finite semigroups generated by semigroups of degree 2. Is membership in V_2 decidable? Does

V_2 have a finite basis of profinite identities?

Bonus from preparing this talk:

Theorem

*V_2=Semilattice*Groups*Right Regular Bands*Groups*

Generalizations and Open Problems

For degree 1, a Theorem of Schein gives a finite basis of implications for the collection of semigroups that have a faithful representation by degree 1 functions and the famous Theorem of Ash shows that the variety generated by semigroups of degree 1 is the variety of finite semigroups whose idempotents commute.

All of these questions can also be asked for the collection of all finite semigroups that have faithful degree k representations for all $k > 1$.

References

S. Margolis and J. Rhodes. Degree 2 transformation semigroups: Foundations and Structure, To Appear, IJAC, Special Issue on Papers from the Sandgal19 Conference.

As a Special Offer for Listeners to this Lecture: Available Free of Charge from the Speaker!!! (Or from Research Gate or ArXiv).

S. Margolis and J. Rhodes. Degree 2 transformation semigroups: Complexity and examples. In preparation.

References

S. Margolis, J. Rhodes, and P. Silva. On the Wilson monoid of a pairwise balanced design. J. Alg. Comb., To appear, 2021. This is a paper on continuous functions in our sense on

matroids, block designs and boolean representable simplicial complexes.

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proof that ts of degree k have complexity at most k and the connection of continuity to translational hulls.

References

J. H. Dinitz and S. W. Margolis. Continuous maps on block designs. Ars Combin., 14:21(45), 1982.

J. H. Dinitz and S. W. Margolis. Continuous maps in finite projective space. In Proceedings of the thirteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1982), volume 35, pages 239(244), 1982.

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Three papers on continuous functions on block designs.

Unsolicited Advice

Write down some graphs. Compute their monoids of continuous functions. Say something interesting...

Wishes

STAY SAFE AND HEALTHY DEAR FRIENDS!!!