Universal Theory of Pregroups

and Partially Commutative Groups

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October 31, 2007

Outline

Partially Commutative Groups

A group G which has a finite presentation $\langle X|R\rangle$ where

$$R \subseteq \{[x,y] : x,y \in X\}$$

is called a *partially commutative* group.

G corresponds to a graph Γ with vertices X and and edge joining x and y if and only if $[x,y]=1$ in G.

G will from now on be a partially commutative group with graph Γ.

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Languages

A language $\mathcal{L}$ with signature (C, F, R) consists of

- a set of constant symbols C;
- a set of function symbols F, each with a positive integer;
- a set of relation symbols R, each with a positive integer;

e.g.

- language of groups $\mathcal{L}_{\mathcal{G}}$: constant 1, unary function $^{-1}$, binary function $\cdot$;
- language of graphs: one binary relation symbol R.

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Formulae

Formulae of $\mathcal{L}$ are built up inductively from

- the symbols of $\mathcal{L}$, $=$, variables $x_1, x_2, \ldots$
- logical symbols $\neg$, $\wedge$, $\vee$, $\to$,
- and quantifiers $\forall$, $\exists$.

In $\mathcal{L}_{\mathcal{G}}$ atomic formulae have the form $w_1 = w_2$, where w_i is an element of the free group on the variables,

and if ϕ, ψ are formulae then so are $\neg\phi$, $\phi \vee \psi$, $\phi \wedge \psi$,

and $\exists x \phi$ and $\forall x \phi$.

- for example $xyzx^{-1}y^{-1}z^{-1} = 1$
- $\forall x(x^n \neq 1 \vee x = 1)$
- $\forall x \forall y \forall z(x^2y^2z^2 = 1 \to [x, y] = 1 \wedge [x, z] = 1 \wedge [y, z] = 1)$.

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Sentences

A **sentence** of $\mathcal{L}$ is an $\mathcal{L}$-formula with no free variable .

Every sentence is equivalent to one in Prenex Normal Form:

$$Q_1 x_1 \cdots Q_k x_k \psi$$

where $Q_i = \forall$ or $\exists$ and ψ is a quantifier free formula.

A sentence is **existential** if all $Q_i = \exists$ and **universal** if all $Q_i = \forall$.

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Equivalence

If an $\mathcal{L}_{\mathcal{G}}$-sentence ϕ holds in a group G write $G \vDash \phi$.

e.g. If A is Abelian $A \vDash \forall x \forall y([x, y] = 1)$.

The **elementary theory** of G is the set of all $\mathcal{L}_{\mathcal{G}}$-sentences ϕ such that $G \vDash \phi$.

G and H are **elemenarily equivalent**, $G \equiv H$, if they have the same elementary theory.

Existential and **universal** theory and equivalence are defined analagoulsy, replacing sentences with existential or universal sentences.

Write $G \equiv_{\exists} H$ and $G \equiv_{\forall} H$.

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If an $\mathcal{L}_{\mathcal{G}}$-sentence ϕ holds in a group G write $G \models \phi$.

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Free and Abelian groups

- Finitely generated non-Abelian free groups are all elementarily equivalent. (Sela and Miasnikov & Kharlampovich.)
- The elementary theory of free groups is decidable (Miasnikov & Kharlampovich).
- Two Abelian groups are elementarily equivalent if and only if their Szmielew invariants are the same.
- Finitely generated torsion-free Abelian groups are elementarily equivalent if and only their ranks are equal.
- The elementary theory of ordered Abelian groups is decidable (Gurevich).
- The positve theory of partially commutative groups is decidable (Casals-Ruiz & Kazachkov).

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Universal and Existential Theory

Let M and N be models of a language $\mathcal{L}$.

$\forall x \phi$ holds iff $\exists x \neg \phi$ does not hold; so

$M \equiv_\forall N$ iff $M \equiv_\exists N$.

$\mathcal{F}(M)$ is the set of isomorphism classes of finite subsets of M.

Lemma

$M \equiv_\exists N$ iff $\mathcal{F}(M) \equiv \mathcal{F}(N)$.

Corollary

1. *All non-Abelian free groups are existentially equivalent.*
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A **pregroup** consists of a set P together with

1. a designtated element 1;
2. an involution $^{-1}$ defined on P;
3. a relation $M \subseteq P \times P \times P$;

such that

1. $\forall x, y, z[(x, y, z) \in M \wedge (x, y, w) \in M \rightarrow z = w]$
2. $\forall x[(x, 1, x) \in M \wedge (1, x, x) \in M]$
3. $\forall x[(x, x^{-1}, 1) \in M \wedge (x^{-1}, x, 1) \in M]$
4. $\forall a, b, c, r, s, x[(a, b, r) \in M \wedge (b, c, s) \in M] \rightarrow [(r, c, x) \in M \leftrightarrow (a, s, x) \in M]$
5. $\forall a, b, c, d, x, y, z[(a, b, x) \in M \wedge (b, c, y) \in M \wedge (c, d, z) \in M \rightarrow [\exists r, s[(a, y, r) \in M \vee (y, d, s) \in M]]$

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1. $\forall x, y, z[(x, y, z) \in M \wedge (x, y, w) \in M \rightarrow z = w]$
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Reduced words

Define $D = \{(a, b) \in P \times P : (a, b, c) \in M \text{ for some } c \in P\}$.

Write ab for c if $(a, b, c) \in M$.

A word of length n: $w = (p_1, \ldots, p_n)$, $p_i \in P$

Reduction: If $(p_i, p_{i+1}) \in D$ reduce to $(p_1, \ldots, p_i p_{i+1}, \ldots p_n)$

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Interleaving

$\mathbf{c} = (c_1, \ldots, c_k)$ and $\mathbf{a} = (a_1, \ldots, a_{k-1})$ words.

If $(c_1, a_1) \in D$, (a_{i-1}^{-1}, c_i) and $(a_{i-1}^{-1} c_i, a_i)$ are in D, for $i = 1, \ldots, k-1$, and $(a_{k-1}, c_k) \in D$

then the *interleaving* $\mathbf{c} * \mathbf{a}$ of $\mathbf{c}$ and $\mathbf{a}$ is

$$(c_1 a_1, a_1^{-1} c_2 a_2, \ldots, a_{k-2}^{-1} c_{k-1} a_{k-1}, a_{k-1} c_k).$$

Define $\mathbf{c} \approx \mathbf{d}$ if and only if $\mathbf{d} = \mathbf{c} * \mathbf{a}$, for some word $\mathbf{a}$.

Stallings: if $\mathbf{c}$ is reduced then so is $\mathbf{c} * \mathbf{a}$ and $\approx$ is an equivalence relation on reduced words over P.

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The *universal group* $U(P)$ of P is the set of equivalence classes of reduced words:

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Characterisation of equivalence

Lemma

Let P be a pregroup and let $(c_1, \ldots, c_m)$ and $(d_1, \ldots, d_n)$ be words. Then $(c_1, \ldots, c_m) \approx (d_1, \ldots, d_n)$ if and only if $m = n$ and

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$r = 1, \ldots m$, and $d_m^{-1} \cdots d_1^{-1} c_1 \cdots c_m = 1$.

Corollary

If Q is a subpregroup of a pregroup P then $U(Q)$ is a subgroup of $U(P)$.

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Pregroups and universal equivalence

The language of pregroups $\mathcal{L}_{\mathcal{P}}$:

- constant 1;
- unary function $^{-1}$;
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Theorem

If $P_1 \equiv_{\exists} P_2$ in $\mathcal{L}_P$ then $U(P_1) \equiv_{\exists} U(P_2)$ in $\mathcal{L}_G$.

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Application to Graphs of Groups

Corollary

Let Γ be a connected, directed graph and let $(\mathcal{G}, \Gamma)$ and $(\mathcal{G}', \Gamma)$ be graphs of groups. Suppose the following conditions hold.

1. $\mathcal{G}(e) = \mathcal{G}'(e)$, *for all edges* $e \in ET$.
2. $\mathcal{G}(v) \equiv_{\exists} \mathcal{G}'(v)$, *for all* $v \in VT$,
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Then $\pi_1(\mathcal{G}, T) \equiv_{\exists} \pi_1(\mathcal{G}', T)$.

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Application to Partially Commutative Groups

T_i a tree:

There are three distinct universal theories.

1. If T_1 has one vertex and T_2 has 2 vertices then $\mathbb{Z} = G(T_1) \equiv_\exists G(T_2) = \mathbb{Z}^2$.
2. If T_1 and T_2 have diameter 2 then $G(T_i) = \mathbb{F}_{n_i} \times \mathbb{Z}$ and $G(T_1) \equiv_\exists G(T_2)$.
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2. If T_1 and T_2 have diameter 2 then $G(T_i) = \mathbb{F}_{n_i} \times \mathbb{Z}$ and $G(T_1) \equiv_{\exists} G(T_2)$.
3. If T_1 and T_2 have diameter more than 2 then $G(T_1) \equiv_{\exists} G(T_2)$.

Application to Partially Commutative Groups

T_i a tree:

There are three distinct universal theories.

1. If T_1 has one vertex and T_2 has 2 vertices then $\mathbb{Z} = G(T_1) \not\equiv_\exists G(T_2) = \mathbb{Z}^2$.
2. If T_1 and T_2 have diameter 2 then $G(T_i) = \mathbb{F}_{n_i} \times \mathbb{Z}$ and $G(T_1) \equiv_\exists G(T_2)$.
3. If T_1 and T_2 have diameter more than 2 then $G(T_1) \equiv_\exists G(T_2)$.