Optimality of the Bartlett-Priestley Window

A Tribute to Professor Maurice Priestley

Jingsong Yuan

School of Mathematics, University of Manchester

The Bartlett-Priestley Window

What it is for

- Professor Priestley made many contributions to Time Series.
- One contribution that bears his name is the Bartlett-Priestley window.
- This is used to smooth the peridogram to make it a consistent estimator of the spectrum.
- I was fascinated by spectral analysis because
  - Peridicities would show up in the periodogram, which is essentially the squared magnitude of the DFT of the signal.
  - The DFT values of a stationary signal are largely uncorrelated, making frequency domain analysis easier than time domain.
  - Higher order cumulant properties can be derived making it possible to prove asymptotic normality.
- One day I shall get my head round the DWT.

The Bartlett-Priestley Window

Definition

- The Bartlett-Priestley window is defined as

$$\begin{aligned} W_N(\theta) &= MK_{BP}(M\theta),\ |\theta| \leq \pi \\ &= \begin{cases} \frac{3M}{4\pi}\left(1 - \frac{M^2\theta^2}{\pi^2}\right), & |\theta| \leq \frac{\pi}{M}, \\ 0, & \text{o.w.} \end{cases} \end{aligned}$$

  for time series (d=1), where M is a 'bandwidth' parameter.
- In d-dimensions,

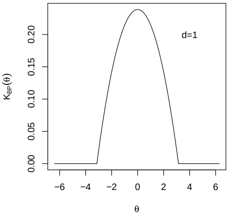
$$W_N(\theta) = M_1 \cdots M_d K_{BP}(M_1\theta_1, \ldots, M_d\theta_d)$$

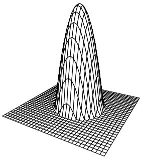
$$K_{BP}(\theta) = \begin{cases} \frac{d(d+2)\Gamma(d/2)}{4\Gamma(1/2)^d\pi^d}\left(1 - \frac{\|\theta\|^2}{\pi^2}\right), & \|\theta\| \leq \pi, \\ 0, & \text{o.w.} \end{cases}$$

The Bartlett-Priestley Window

Properties of $K_{BP}(\theta)$

- Non-negative valued
- Integrates to one
- Peak at $\theta = 0$
- Support on $(-\pi, \pi]^d$





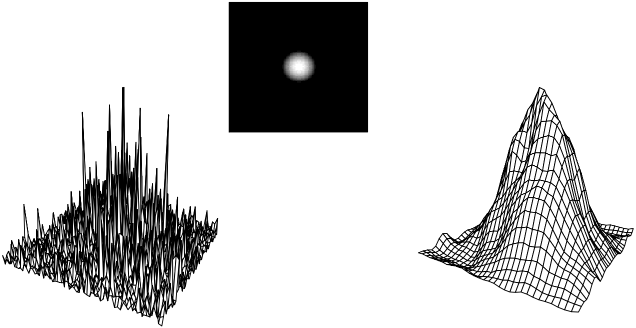
- Quadratic $\Rightarrow$ optimality

The Bartlett-Priestley Window

How it works

$$\hat{f}(\omega) = \int_{(-\pi,\pi]^d} I_N(\theta) W_N(\omega - \theta) \mathrm{d}\omega, \ \omega \in (-\pi, \pi]^d,$$

Weighted average of periodogram values $I_N(\theta)$ near ω.



The Bartlett-Priestley Window

Optimality

- Mean square error (minimised over M):

$$\min_M \mathrm{E}[(\hat{f}(\omega) - f(\omega))^2] \approx \left(1 + \frac{d}{4}\right)(2\pi)^{\frac{4d}{d+4}} d^{-\frac{d}{d+4}}$$

$$\left[(1+\eta(2\omega))^2 f(\omega)^4 \prod_{i=1}^{d} f_{ii}^{(2)}(\omega)\right]^{\frac{2}{d+4}}$$

$$\Bigg\{\underbrace{\int_{R^d} K(\theta)^2 \mathrm{d}\theta \left[\int_{R^d} \|\theta\|^2 K(\theta)\mathrm{d}\theta\right]^{\frac{d}{2}}}_{C(K)}\Bigg\}^{\frac{4}{d+4}}$$

Smaller is better

- $K_{BP}(\theta)$ has smallest $C(K)$ among all $K(\theta)$ that is non-negative and integrates to one.

The Bartlett-Priestley Window

Proof of optimality

For any $K(\theta) \geq 0$, $\int_{R^d} K(\theta)\mathrm{d}\theta = 1$, choose $\delta > 0$ so that

$$\int_{R^d} \|\theta\|^2 K_\delta(\theta)\mathrm{d}\theta = \int_{R^d} \|\theta\|^2 K_{BP}(\theta)\mathrm{d}\theta \quad \text{(Same bias)}$$

where $K_\delta(\theta) = \delta^d K(\delta\theta_1, \ldots, \delta\theta_d)$. Then $\int_{R^d} K_\delta(\theta)\mathrm{d}\theta = 1$

$$\Rightarrow \int_{R^d} \left(1 - \frac{\|\theta\|^2}{\pi^2}\right) K_{BP}(\theta)\mathrm{d}\theta = \int_{R^d} \left(1 - \frac{\|\theta\|^2}{\pi^2}\right) K_\delta(\theta)\mathrm{d}\theta$$

$$\Rightarrow \int K_{BP}(\theta)^2\mathrm{d}\theta \leq \int K_{BP}(\theta)K_\delta(\theta)\mathrm{d}\theta \leq \sqrt{\int K_{BP}(\theta)^2\mathrm{d}\theta \int K_\delta(\theta)^2\mathrm{d}\theta}$$

$$\Rightarrow \int_{R^d} K_{BP}(\theta)^2\mathrm{d}\theta \leq \int_{R^d} K_\delta(\theta)^2\mathrm{d}\theta \Rightarrow C(K_{BP}) \leq \underline{C(K_\delta) = C(K)}.$$

(Smaller variance) (Smaller MSE)

Scale Invariance of C(K)

Further details

- $C(K)$ is MSE miminised over M which is a vector of scale parameters. You can expect that $C(K_\delta) = C(K)$ for any $\delta > 0$.
- "More specifically" (as Professor Priestley would write),

$$\int_{R^d} K_\delta(\theta)\mathrm{d}\theta = \int_{R^d} \delta^d K(\delta\theta_1, \ldots, \delta\theta_d)\mathrm{d}\theta = \int_{R^d} K(\theta)\mathrm{d}\theta = 1,$$

$$\int_{R^d} \|\theta\|^2 K_\delta(\theta)\mathrm{d}\theta = \int_{R^d} \|\theta\|^2 \delta^d K(\delta\theta_1, \ldots, \delta\theta_d)\mathrm{d}\theta = \frac{1}{\delta^2}\int_{R^d} \|\theta\|^2 K(\theta)\mathrm{d}\theta,$$

$$\int_{R^d} K_\delta(\theta)^2\mathrm{d}\theta = \int_{R^d} \delta^{2d} K(\delta\theta_1, \ldots, \delta\theta_d)^2\mathrm{d}\theta = \delta^d \int_{R^d} K(\theta)^2\mathrm{d}\theta,$$

$$C(K_\delta) = \delta^d \int_{R^d} K(\theta)^2\mathrm{d}\theta \left[\frac{1}{\delta^2}\int_{R^d} \|\theta\|^2 K(\theta)\mathrm{d}\theta\right]^{\frac{d}{2}} = C(K).$$