Commemoration meeting in honour of

Professor Maurice Priestley

Manchester University

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Continuous Time Spectral Analysis for Systems Response Estimation

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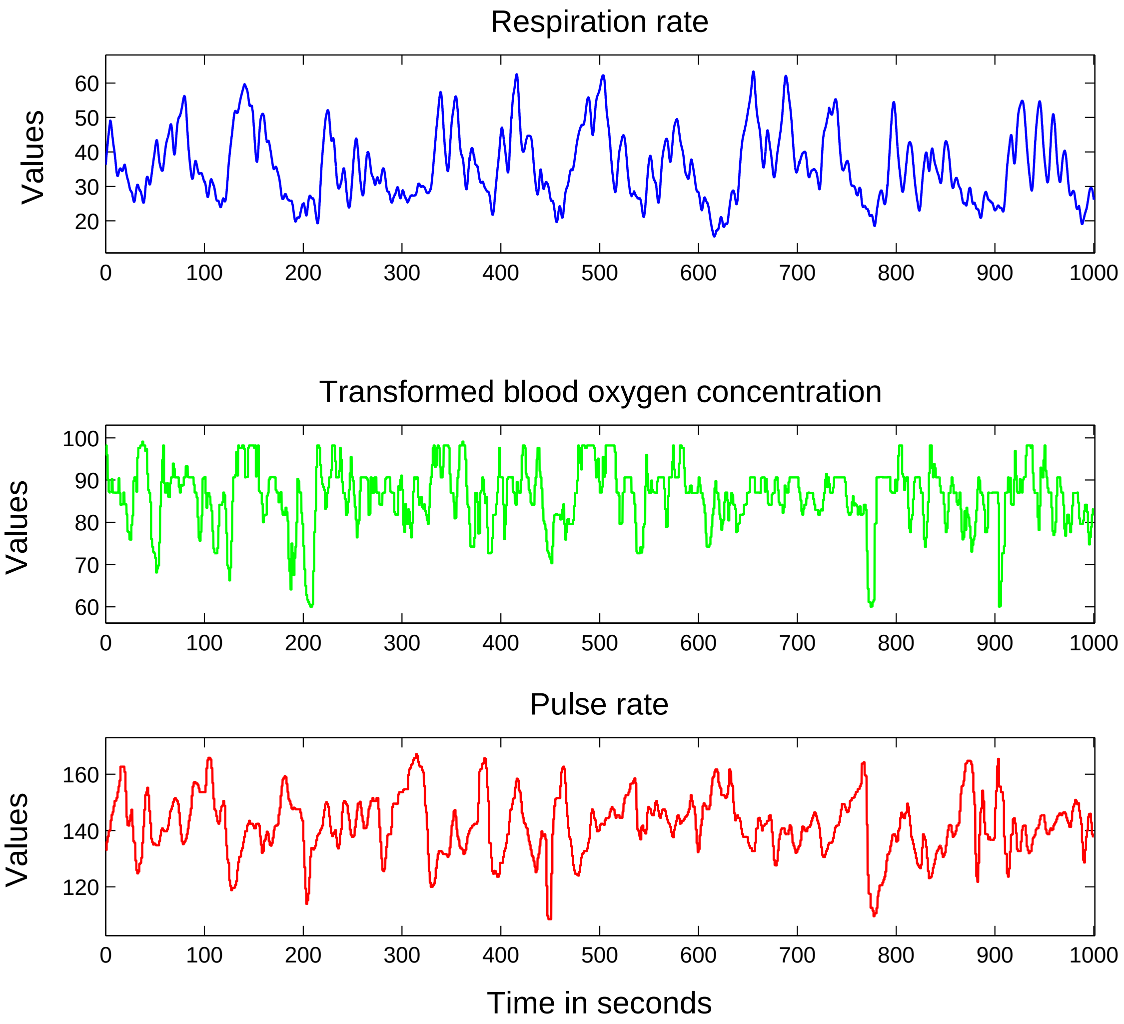
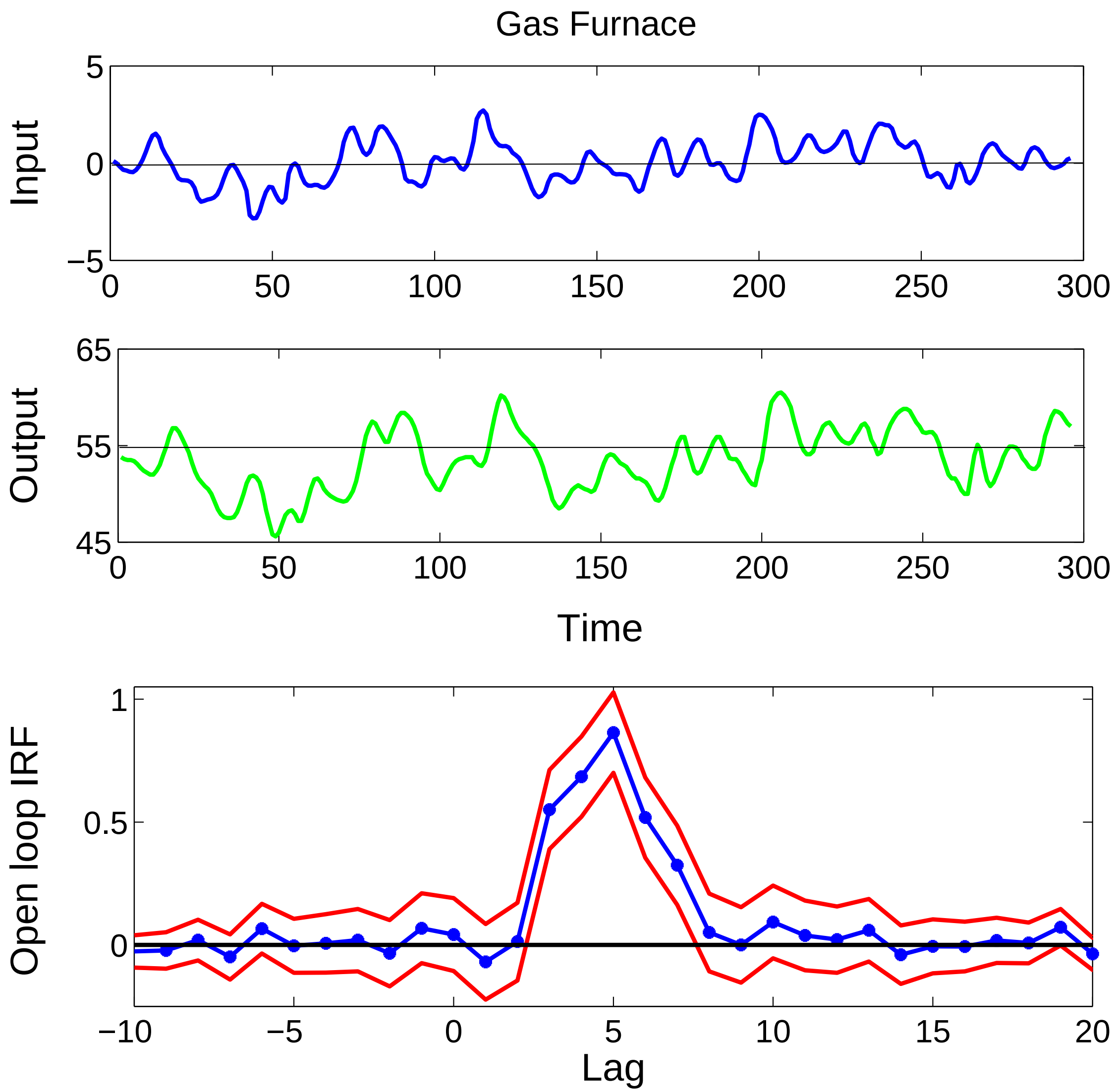
- Motivating data set are three indicators of the respiratory system of pre-term infants.
- Shown here over a period of 1000 seconds recorded at 1/10th second intervals

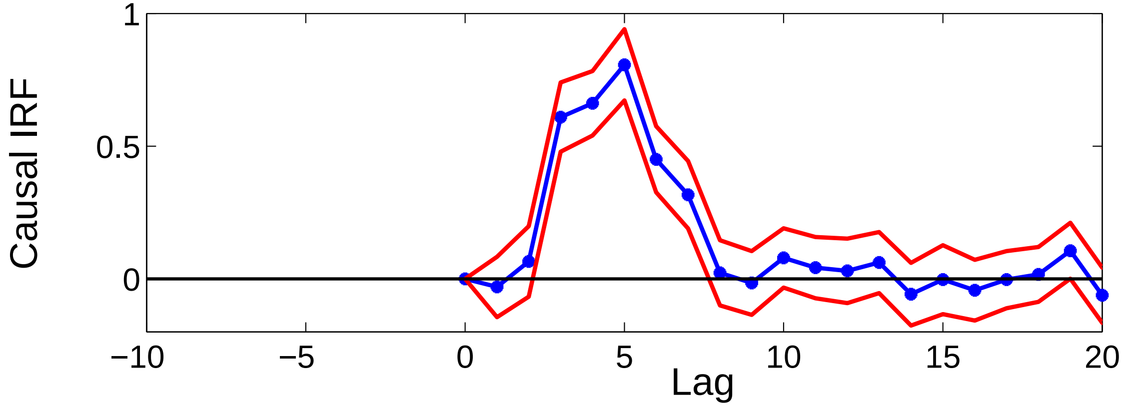


Though in truth discrete, the sampling frequency is so high we treat them as continuous.

- We would like to characterize the series by the system impulse response
- For discrete series we can apply classical spectral analysis
- Only in an open loop system does this estimate a causal response



- A one sided (causal) estimate of the IRF of an open loop system can be given following methods of Bhansali and Karavellas (1983) to construct the Wiener filter using univariate spectral factorization.
- For a closed loop system the multivariate spectrum of the series can be factorized to give the innovation representation of the series.
- From this a one-sided estimate of IRF can be constructed - shown for the Gas Furnace example again.



The smoothing bandwidth can be selected using a Final Prediction Error criterion.

- For an open loop system of input x_t and output y_t the frequency response is estimated by the ratio of smoothed cross-spectrum to input spectrum:

$$T(f) = \frac{S_{y\,x}(f)}{S_{x\,x(f)}}$$

and $T(f)_k$ is the time domain response to an input impulse.

- For a closed loop system the bivariate spectrum is factorized as

$$\begin{pmatrix} S_{y\,y}(f) & S_{y\,x}(f) \\ S_{x\,y}(f) & S_{y\,y}(f) \end{pmatrix} = \Psi(f) G\, \overline{\Psi}(f)'$$

corresponding to the innovations representation

$$x_t = \sum_{k=0}^{\infty} \psi_k e_{t-k} \quad \text{with} \quad \psi_k = \Psi(f)_k.$$

- The response of y_t to an innovation impulse in x_t is $\psi_{y\,x,k}$.

- The response of y_t to an impulse in x_t is $-\left\{\Pi_{y\,x}(f)/\Pi_{y\,y}(f)\right\}_k$ where $\Pi(f) = \Psi(f)^{-1}$.

- If $\Psi_{x\,y}(f) \equiv 0$ this reduces to both $\left\{\Psi_{y\,x}(f)/\Psi_{x\,x}(f)\right\}_k$ and $\left\{S_{y\,x}(f)/S_{x\,x}(f)\right\}_k$.

- The spectral power in the respiration series resides at very low frequencies.
- To avoid this, high frequency data is commonly smoothed and sub-sampled.
- How does the discrete innovation representation correspond to the continuous?

$$x_t = \sum_{k=0}^{\infty} \psi_k e_{t-k} \qquad \Leftrightarrow \qquad x(t) = \int_0^{\infty} \psi(h) dB(t-h)$$

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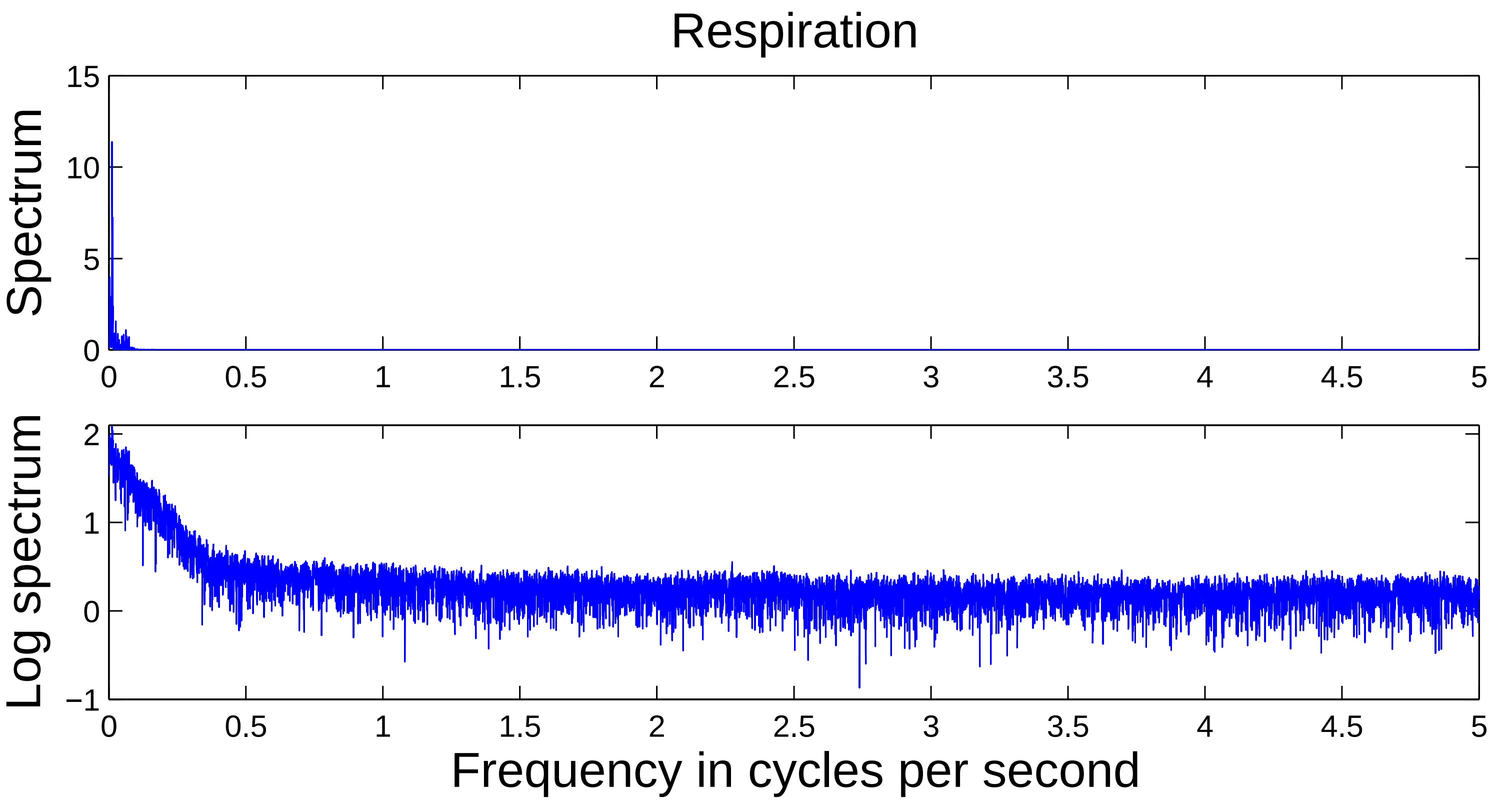
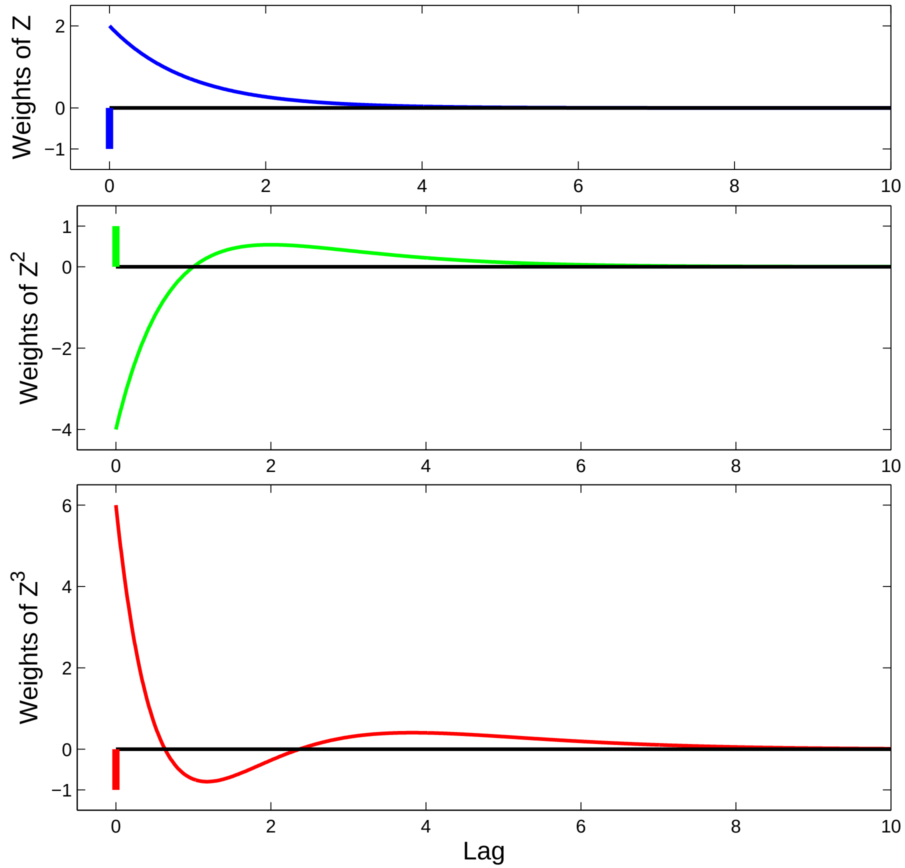


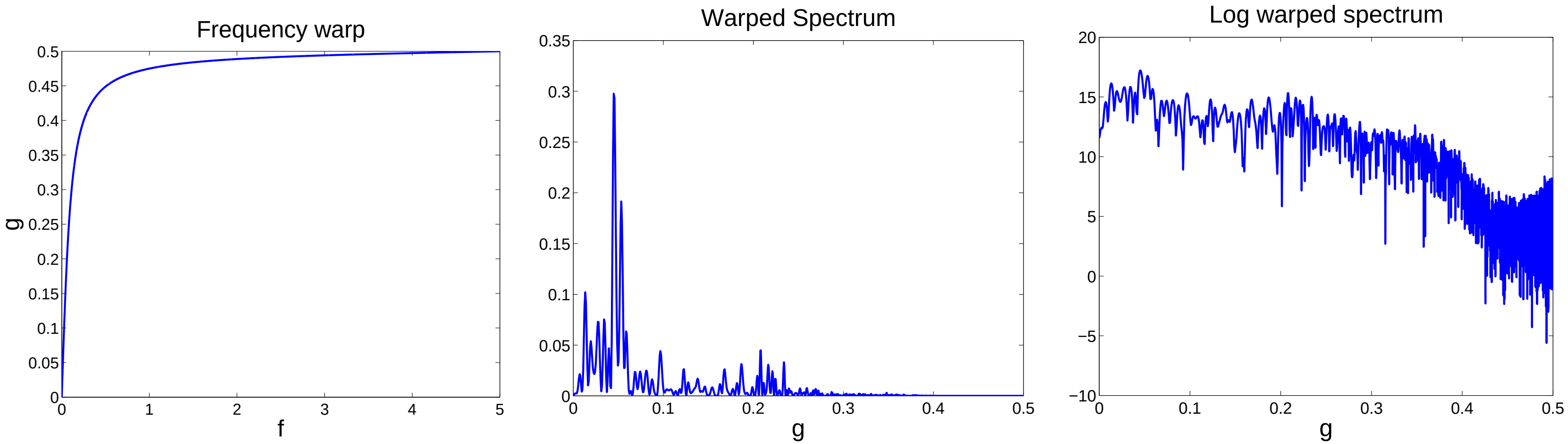
This may be sensitive to low lag correlation and the high frequency spectrum.

- Use a discrete basis of continuous time series $x(t)$ to construct discrete $X_{-k} = Z^k x(t)$
- $Z = \frac{\kappa - s}{\kappa + s}$ where κ is a rate constant, chosen by inspection of spectrum as 0.4
- $s = 2\pi i f$ or Laplace operator gives basis weighting illustrated for $\kappa = 1$.

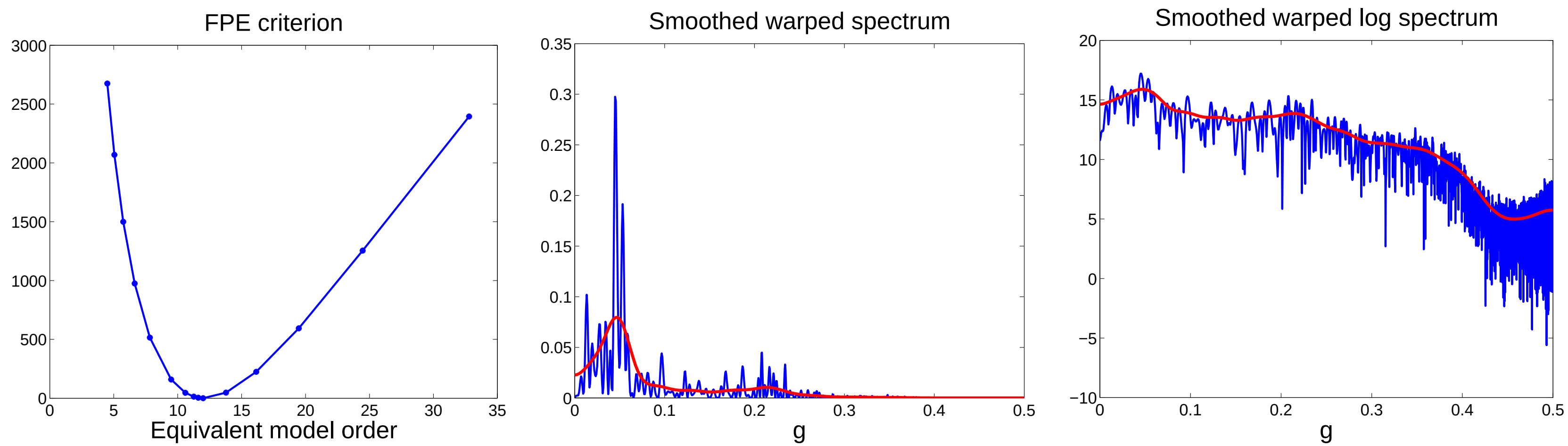


- Z is like a back-shift of $2/\kappa$ at low frequencies - no information lost as in sub-sampling.
- Implicit in Wiener's solution of continuous time prediction - explicit in Doob p583.

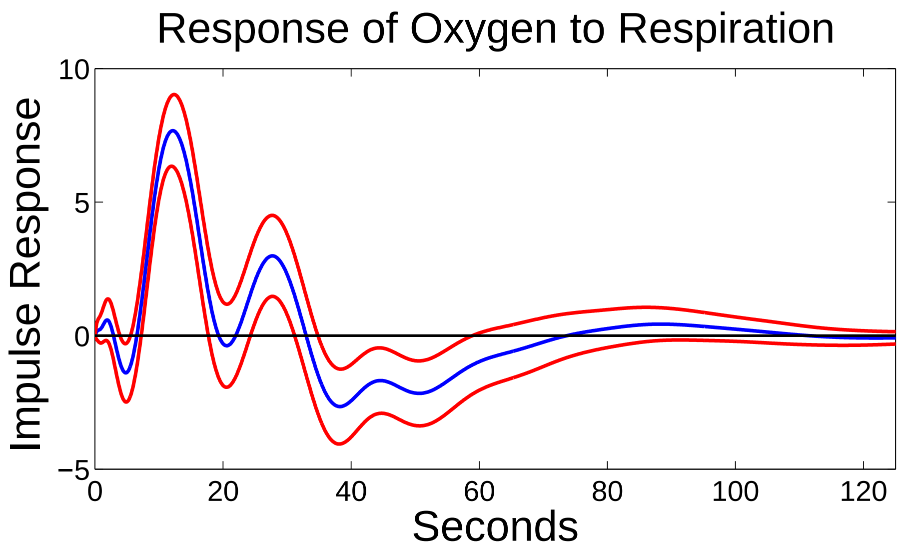
- Spectrum of X_k related to that of $x(t)$ by $\boxed{S_X(g)dg = S_x(f)df}$
  where $g(f) = (1/\pi)\arctan(2\pi f/\kappa)$ is a $\boxed{\text{frequency warp}}$
  - an operation of well established and contemporary use in speech signal processing.

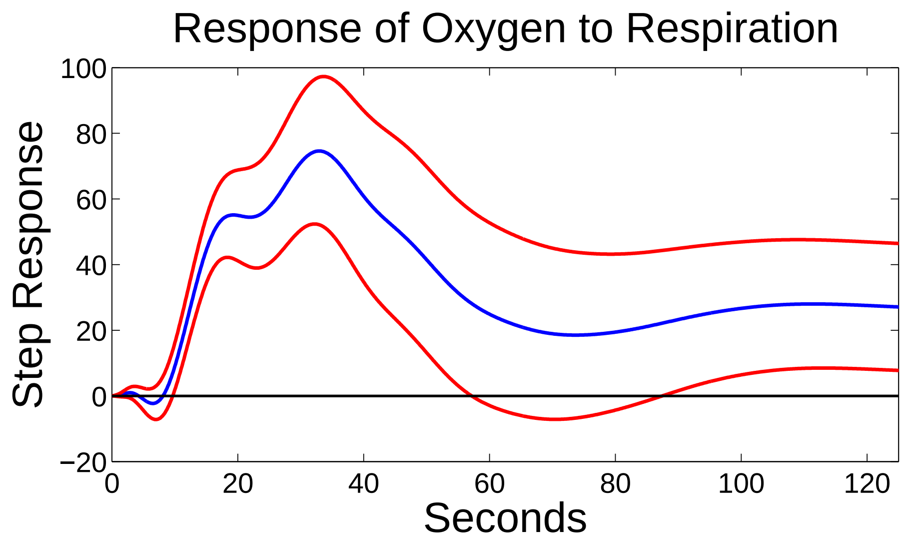


- Apply lag window smoothing to warped spectrum - bandwidth chosen by FPE of X_k;



- Factorize $S_X(g) = R(g)\overline{R(g)}'$
- Undo warp $T(f) = R(g(f))\sqrt{2\kappa}/(\kappa + 2\pi i f)$
- Normalize to $\Psi(f) = T_0^{-1}T(f)$
- Transform to $\psi(h)$ for impulse response





These responses may be used to characterize the functionality of the infant respiration.

Some further technical aspects

- The frequency warp $g(f) = (1/\pi)\arctan(2\pi f/\kappa)$ is derived from

$$\exp(2\pi i g) = \frac{\kappa - 2\pi i f}{\kappa + 2\pi i f}$$

- The warp has Jacobian

$$df = \frac{k}{2}\sec^2(\pi g)dg \quad \text{or} \quad dg = \frac{2\kappa}{\kappa^2 + (2\pi f)^2} df = \left|\frac{\sqrt{2\kappa}}{\kappa + 2\pi i f}\right|^2 df$$

  which is the spectrum of a CAR(1) with parameter κ.

- For bandwidth selection the deviance or $-2\log$ likelihood term in the FPE is given by

$$\kappa T \int_{-0.5}^{0.5} \log\det S_X^*(g)dg$$

  where T is the record length and $S_X^*(g)$ the sample warped spectrum.

- The penalty term for bandwidth selection is given for the smoothing lag window w_k by

$$2m^2 \sum_{k=1}^{M} w_k^2$$

- The standard error limits are derived by simulation of the sample auto and cross-covariances of the sample warped spectrum of a multivariate CAR(1) residual series with unit variance. These have variance $1/(\kappa T)$ and values at lags $k, k+1$ have correlation $1/2$.