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Bootstrap Methods in Time Series – In memoriam of Professor Maurice Bertram Priestley

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Overview

Overview

Stationary Processes

Let $\mathbf{X} = (X_t : t \in \mathbb{Z})$ be a linear process, that is

$$X_t = \sum_{j=-\infty}^{\infty} \alpha_j \varepsilon_{t-j} \ , t \in \mathbb{Z},$$

(ε_t) i.i.d. $(0, \sigma^2)$, $E\varepsilon_t^4 < \infty$ and $\sum_j |j|^{1/2} |\alpha_j| < \infty$.

Autocovariance $\gamma(h) = Cov(X_t, X_{t+h})$ and spectral density

$$f(\lambda) = (2\pi)^{-1} \sum_{h=-\infty}^{\infty} \gamma(h) e^{-i\lambda h} \ .$$

Based on observations $X_1, X_2, \ldots, X_n$ we have the periodogram

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{-i\lambda t} \right|^2 \;, \lambda \in [-\pi, \pi] \,,$$

and we intend to consider statistics of the form

$$\int_{-\pi}^{\pi} \phi(\lambda) \, I_n(\lambda) \, d\lambda \;,$$

for suitable functions ϕ auf $[-\pi, \pi]$.

Examples are:

$$\widehat{\gamma}(h) = \int_{-\pi}^{\pi} \cos(\lambda h) I_n(\lambda)\, d\lambda = \frac{1}{n} \sum_t X_t X_{t+h} \ ,$$

or, for $\phi(\lambda) = \mathbf{1}_{[0,x]}(\lambda)$, the *integrated* periodogram

$$F_n(x) = \int_0^x I_n(\lambda)\, d\lambda \ .$$

Essential Properties of the Periodogram

- $E[I_n(\lambda_j)] = f(\lambda_j) + \mathcal{O}(n^{-1})$ und

  $$\mathrm{Cov}[I_n(\lambda_j), I_n(\lambda_k)] = \begin{cases} f^2(\lambda_j) + \mathcal{O}(n^{-1}) & 0 < \lambda_j = \lambda_k < \pi \\ \frac{\eta_4 f(\lambda_j) f(\lambda_k)}{n} + o(n^{-1}) & 0 < \lambda_j \neq \lambda_k < \pi \end{cases}$$

  where $\eta_4 = \kappa_4/\sigma^4$ and $\kappa_4 = E\varepsilon_t^4 - 3\sigma^4$.

  $\lambda_j = 2\pi j/n$, $j \in \{-[(n-1)/2], -[(n-1)/2]+1, \ldots, [n/2]\}$ denote the Fourier frequencies.

- We have $I_n(\lambda_j) \approx f(\lambda_j) \cdot U_j$, U_j exponentially distributed and independent.
  But: Periodogram ordinates $I_n(\lambda_j)$ are asymptotically independent, only.

Under suitable assumptions we have that

$$M_n := \sqrt{n} \left(\int_{-\pi}^{\pi} \phi(\lambda) I_n(\lambda)\, d\lambda - \int_{-\pi}^{\pi} \phi(\lambda) f(\lambda)\, d\lambda \right)$$

converges in distribution towards $\mathcal{N}(0, \sigma_\phi^2)$ with

$$\sigma_\phi^2 = 2\pi \int_{-\pi}^{\pi} \phi^2(\lambda) f^2(\lambda)\, d\lambda + \eta_4 \left(\int_{-\pi}^{\pi} \phi(\lambda) f(\lambda)\, d\lambda \right)^2 .$$

Remark: Second summand is due to finite sample dependence of the periodogram ordinates! If we want to mimic the distribution of integrated periodograms asymptotically correct we have (to a sufficient extend) to mimic the vanishing dependence between periodogram ordinates.

A simple frequency Bootstrap

Idea:

1: Generate exponentially distributed random variables $U_1^*, U_2^*, \ldots$

2: Replace unknown spectral density through $\widehat{f}(\lambda_j)$

3: Generate pseudo integrated periodograms ordinates according to

$$I_n^*(\lambda_j) = \widehat{f}(\lambda_j) \cdot U_j^*$$

Cf. Hurvich and Zeger (1987), Härdle and Franke (1992), Dahlhaus and Janas (1996), ...

A negative result!

The simple frequency Bootstrap leads for

$$M_n^* = \sqrt{n}\left(\int_{-\pi}^{\pi} \phi(\lambda) I_n^*(\lambda)\, d\lambda - \int_{-\pi}^{\pi} \phi(\lambda) \widehat{f}(\lambda)\, d\lambda\right)$$

to

$$M_n^* \Rightarrow \mathcal{N}\left(0, 2\pi \int_{-\pi}^{\pi} \phi^2(\lambda) f^2(\lambda)\, d\lambda\right) ,$$

i.e., the simple frequency based approach does not work in general!

- **Notice,** that the second term in the variance

$$\sigma_\phi^2 = 2\pi \int_{-\pi}^{\pi} \phi^2(\lambda) f^2(\lambda) + \eta_4 \Big(\int_{-\pi}^{\pi} \phi(\lambda) f(\lambda) d\lambda \Big)^2.$$

of the limiting distribution of M_n disappears if:

1. $\eta_4 = 0$, for instance if $\mathbf{X}$ is a Gaussian process,

   and/or

2. the function ϕ is such that $\int_{-\pi}^{\pi} \phi(\lambda) f(\lambda) d\lambda = 0$.

- Thus apart from these two cases and because the simple frequency bootstrap procedure generates independent pseudo-periodogram ordinates, it **fails** in general in approximating correctly the asymptotic distribution of spectral means $M_n(I_n, \phi)$.

- **However,** there are important examples of statistics for which the simple frequency bootstrap procedure works.
- Such examples include the class of **ratio-statistics** and the class of **nonparametric spectral density estimators**.
- **Ratio statistics** are defined as

$$R_n(I_n, \phi) = \int_{-\pi}^{\pi} \phi(\lambda) I_n(\lambda) d\lambda \Big/ \int_{-\pi}^{\pi} I_n(\lambda) d\lambda.$$

One important example in this class of statistics are sample autocorrelations:

$$\widehat{\rho}(k) = \widehat{\gamma}(k)/\widehat{\gamma}(0) = \int_{\pi}^{\pi} \cos(\lambda k) I_n(\lambda) d\lambda \Big/ \int_{\pi}^{\pi} I_n(\lambda) d\lambda.$$

A positive result!

Reason: For ratio statistics we have,

$$\sqrt{n}(R(I_n,\phi) - R(f,\phi)) \Rightarrow N(0,\sigma_R^2).$$

where

$$\sigma_R^2 = \left(2\pi \int w^2(\lambda) f^2(\lambda)\right) \Big/ \left(\int f(\lambda) d\lambda\right)^4.$$

The mentioned simple frequency based bootstrap approach works for so-called ratio statistics.

Dahlhaus and Janas (1996).

- **Nonparametric spectral density estimators** can be obtained by smoothing the periodogram, i.e.,

$$\widehat{f}(\lambda) = n^{-1} \sum_j K_h(\lambda - \lambda_j) I_n(\lambda_j),$$

  where $K_h(\cdot) = h^{-1}K(\cdot/h)$, K is a smoothing kernel and h a bandwidth ($h = h(n) \to 0$, $nh^2 \to \infty$ as $n \to \infty$).

- For the nonparametric estimator $\widehat{f}$ we obtain

$$\sqrt{nh}\left(\widehat{f}(\lambda) - E\,\widehat{f}(\lambda)\right) \Rightarrow N\left(0, f^2(\lambda)(2\pi)^{-1} \int K^2(u)du\right).$$

  Thus the weak and of order n^{-1} vanishing dependence of the periodogram ordinates **does not show-up** in the asymptotic distribution of nonparametric spectral density estimators.

- **QUESTION:** Can we develop a periodogram bootstrap procedure, which is valid for a more general class of statistics, e.g. for general spectral means?
- Clearly, for this, the bootstrap procedure should mimic correctly the weak dependence structure of the periodogram ordinates.
  Janas and Dahlhaus (1994) considered a *post resampling* modification of the standard frequency based bootstrap that works for spectral means.
- Another attempt in this direction is the **Autoregressive Aided Periodogram Bootstrap** proposed by Kreiss and Paparoditis (2003):

  Idea: Use a **hybrid** bootstrap approach by combining a parametric time domain with a nonparametric frequency domain procedure.

Parametric Aided Periodogram Bootstrap

I. Fit an autoregression of order P to data
$X_1, \ldots, X_n \rightsquigarrow \widehat{a}_1(P), \ldots, \widehat{a}_P(P),\ \widehat{\sigma}(P)$.
Residuals: $\widehat{\varepsilon}_t = X_t - \sum_{\nu=1}^{P} \widehat{a}_\nu(P) X_{t-\nu}$, $t = P+1, \ldots, n$
$\widehat{F}_n^c$: centred empirical distribution of standardized residuals

II. Simulate $X_1^+, \ldots, X_n^+$ according to

$$X_t^+ = \sum_{\nu=1}^{P} \widehat{a}_\nu(P) X_{t-\nu}^+ + \widehat{\sigma}(P) \cdot \varepsilon_t^+ ,$$

(ε_t^+) i.i.d. according to $\widehat{F}_n^c$.
$\widehat{f}_{AR}$ spectral density estimator from AR-fit, i.e. the spectral density of (X_t^+).

Parametric Aided Periodogram Bootstrap

III. Compute from $X_1^+, \ldots, X_n^+$:

$$I_n^+(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t^+ e^{-i\lambda t} \right|^2 \ , \ 0 \leq \lambda \leq \pi \ ,$$

IV. and a nonparametric estimator of the correction term $q = f/f_{AR}$

$$\widehat{q}(\lambda) = \frac{1}{n} \sum_{j=-N}^{+N} K_h(\lambda - \lambda_j) \frac{I_n(\lambda_j)}{\widehat{f}_{AR}(\lambda_j)} \ .$$

$K : [-\pi, \pi] \rightarrow [0, \infty)$ kernel, $K_h(\cdot) = 1/h \ K(\cdot/h)$ and $h > 0$ bandwidth.

V. The Bootstrap Periodogram I_n^* is defined as follows

$$I_n^*(\lambda) = \widehat{q}(\lambda) I_n^+(\lambda) \ , \ 0 \leq \lambda \leq \pi.$$

Parametric Aided Periodogram Bootstrap

We obtain for (cf. K. and Paparoditis (2003))

$$\sqrt{n}\left(\int_0^\pi \phi(\lambda) I_n^*(\lambda)\, d\lambda - \int_0^\pi \phi(\lambda)\widetilde{f}(\lambda)\, d\lambda\right) \text{ , where } \widetilde{f} = \widehat{q}\cdot\widehat{f}_{AR} \;:$$

Theorem

For each fixed P (in probability)

$$\mathcal{L}\left(\sqrt{n}\left(\int_0^\pi \phi(\lambda) I_n^*(\lambda)\, d\lambda - \int_0^\pi \phi(\lambda)\widetilde{f}(\lambda)\, d\lambda\right)\right)$$

$$\Rightarrow \;\; \mathcal{N}\left(0, 2\pi\int_0^\pi \varphi^2 f^2\, d\lambda + \eta_4(p)\left(\int_0^\pi \phi f\, d\lambda\right)^2\right)$$

where $\eta_4(p) = \frac{E(X_t - \sum_{\nu=1}^{p} a_\nu(p) X_{t-\nu})^4}{\sigma(P)^4} - 3 \neq \eta_4$ *(in general) !!!*

Parametric Aided Periodogram Bootstrap for Locally Stationary Time Series

The same methodology works for locally stationary time series.
(cf. Paparoditis and Sergides (2008)).

Locally Stationary Processes

Consider triangular array $\{X_{1,n}, \ldots, X_{n,n}\}$

$$X_{t,n} = \sum_{j=-\infty}^{\infty} \alpha_{t,n}(j)\varepsilon_{t-j} \ ,$$

for (ε_t) i.i.d. (0,1) with $E\varepsilon_t^4 < \infty$, continuously differentiable functions $\alpha(\cdot, j) : [0,1] \to \mathbb{R}$ as well as a sequence $(\ell(j) : j \in \mathbb{Z})$ such that

$$\sup_u |\alpha(u,j)| \leq \frac{K}{\ell(j)} \qquad \sum_{j \in \mathbb{Z}} |j| |\ell(j)|^{-1} < \infty \ ,$$

and

$$\sup_{1 \leq t \leq n} \left| \alpha_{t,n}(j) - \alpha(\frac{t}{n}, j) \right| \leq \frac{K}{n\ell(j)} \ .$$

The generated process is denoted locally stationary (cf. Priestley (1965, 1981, 1988), Dahlhaus (1996), ...).

Locally Stationary Processes

We have a family of stochastic processes

$$X_t(u) = \sum_{j=-\infty}^{\infty} \alpha_j(u)\varepsilon_{t-j} \ , \ u \in [0,1] \ ,$$

for (ε_j) i.i.d. (0,1) with $E\varepsilon_t^4 < \infty$ and continuously differentiable functions $\alpha_j(\cdot) : [0,1] \to \mathbb{R}$.

In a sense $X_t(u)$ describes the behavior of the underlying observations $X_{t,n}$ as long as $t/n \sim u$.

Locally Stationary Processes

Local spectral density

$$f(u,\lambda) = \frac{1}{2\pi}|A(u,e^{-i\lambda})|^2 , \quad u \in [0,1],\ \lambda \in (-\pi,\pi] ,$$

where $A(u,z) = \sum_{j\in\mathbb{Z}} \alpha(u,j)z^j$.
Local periodogram

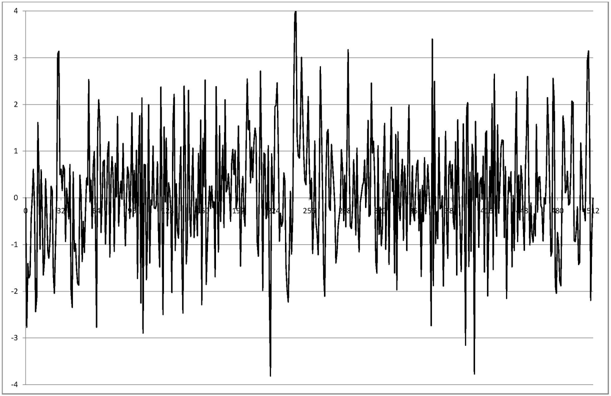
$$I_N(u,\lambda) = \frac{1}{2\pi N}\left|\sum_{t=1}^{N} X_{t+[un]-N/2-1,n}e^{-i\lambda t}\right|^2 ,$$

$\lambda \in [0,\pi]$ and time window $0 < N \ll n$.
One may consider statistics similar as above, namely

$$L_n(u,\phi) = \int_{-\pi}^{\pi} \phi(\lambda) I_N(u,\lambda)\, d\lambda .$$

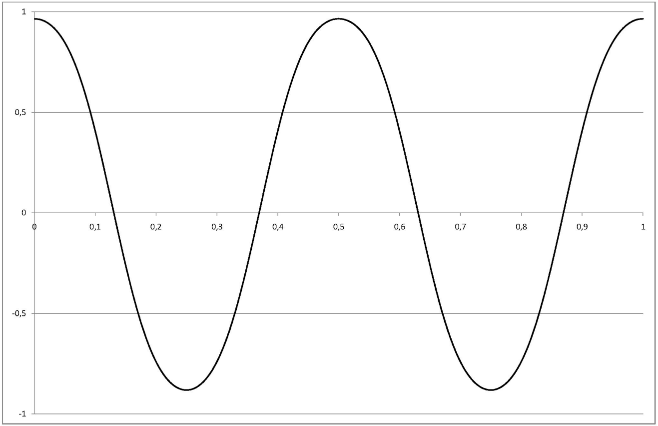
Simulation

Simulated time series (sample size $n = 512$) of first order moving average type: $X_{t,n} = 1.1\cos(1.5 - \cos(4\pi t/n))\varepsilon_{t-1} + \varepsilon_t$



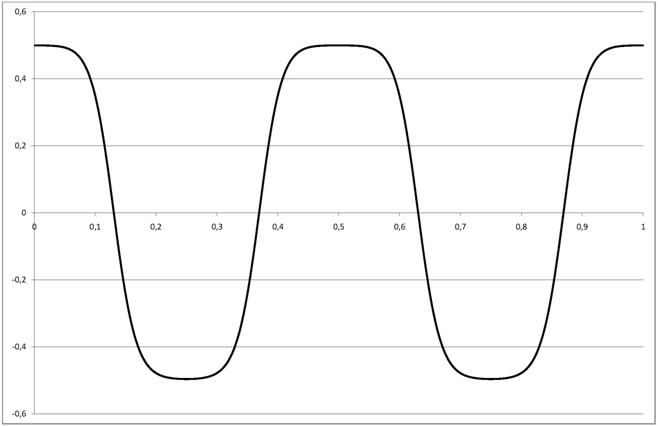
Simulation

Moving average parameter $\alpha(u,1) = 1.1\cos(1.5 - \cos(4\pi u))$



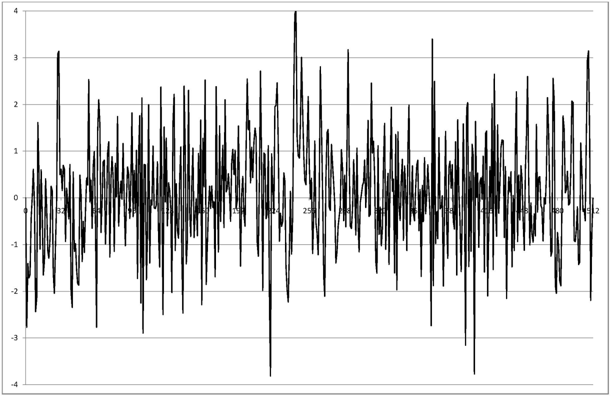
Simulation

Autocorrelation function $\varrho(u, 1)$ (lag 1) as function of u



Simulation

Simulated time series (sample size $n = 512$) of first order moving average type: $X_{t,n} = 1.1\cos(1.5 - \cos(4\pi t/n))\varepsilon_{t-1} + \varepsilon_t$



The Preperiodogram

The Bootstrap proposals considered so far only work in the frequency domain.
But there are relevant statistics in the field of time series analysis which cannot be represented in the frequency domain but essentially need observations in the time domain.

One important tool in the analysis of locally stationary processes is the so-called Preperiodogramm

$$J_n(\frac{t}{n},\lambda) = \frac{1}{2\pi} \sum_{k:1\leq[t+1/2\pm k/2]\leq n} X_{[t+1/2+k/2],n} X_{[t+1/2-k/2],n} e^{-i\lambda k} \ ,$$

cf. Neumann and von Sachs (1997), Nason, Kroisandt and von Sachs (2000), . . .

Motivation

To motivate the bootstrap procedure consider once again the stationary case.

Recall that we can recover $X_{1,n},\ldots,X_{n,n}$ from the finite Fourier transform, i.e.,

$$X_{t,n}=\sqrt{\frac{2\pi}{n}}\sum_{j} J_{X,n}(\lambda_j)e^{it\lambda_j}\,,\qquad t=1,2,\ldots,n,$$

where $J_{X,n}(\lambda_j)=(2\pi n)^{-1/2}\sum_{t=1}^{n}X_t e^{it\lambda_j}$

is the discrete Fourier transform of $X_{1,n},\ldots,X_{n,n}$.

$|J_{X,n}(\lambda_j)|^2$ coincides with the periodogram.

Motivation

Using the approximate identity $|J_{X,n}(\lambda_j)|^2 \approx f(\lambda_j) \cdot |J_{\varepsilon,n}(\lambda_j)|^2$, which is true for linear processes, cf. Brockwell and Davis (1991), we obtain

$$X_{t,n} \approx \sqrt{\frac{2\pi}{n}} \sum_j f^{1/2}(\lambda_j) J_{\varepsilon,n}(\lambda_j) e^{it\lambda_j} ,$$

where f denotes the spectral density of the underlying stationary linear process and $J_{\varepsilon,n}(\lambda_j) = (2\pi n)^{-1/2} \sum_{t=1}^{n} \varepsilon_t e^{it\lambda_j}$ is the finite Fourier transform of innovations $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.

Motivation

In analogy we suggest for the locally stationary case to consider the following approximation to $X_{t,n}$,

$$\widetilde{X}_{t,n} = \sqrt{\frac{2\pi}{n}} \sum_j f^{1/2}\left(\frac{t}{n}, \lambda\right) J_{\varepsilon,n}(\lambda_j) e^{it\lambda_j} \, .$$

In this expression we take care of the locally stationary situation in which the spectral density additionally to frequency, also depends on t.

Assumption

For the bootstrap procedure we assume that we have an estimator $\widehat{f}$ of the time varying spectral density satisfying

$$\sup_{u,\lambda} \left| \widehat{f}^{(r)}(u,\lambda) - f^{(r)}(u,\lambda) \right| \to 0 \text{ in probability, for } r = 0, 1.$$

and that we have *some* bootstrap replicates $\varepsilon_1^*, \ldots, \varepsilon_n^*$ of the innovations ε_t.

Even if the innovations ε_t are not observable, we can imitate their essentials by means of pseudo-innovations, that are generated such that the first, second and fourth order moment structure of the true innovations ε_t's are correctly mimicked.

Estimation fourth order cumulant of innovations

We suggest a nonparametric estimator of the unknown rescaled fourth order cumulant $\eta_4 = E\,\varepsilon_t^4/\sigma^4 - 3$, borrowing ideas from the stationary set-up discussed in Grenander and Rosenblatt (1957).

For $X_t(u) = \sum_{j=-\infty}^{\infty} \alpha_j(u)\varepsilon_{t-j}$, $u \in [0,1]$, we can verify

$$\mathrm{Cov}(X_t^2(u), X_{t+k}^2(u)) = \kappa_4 \sum_{j=-\infty}^{\infty} \alpha_j^2(u)\alpha_{j+k}^2(u) + 2\,\mathrm{Cov}^2(X_t(u), X_{t+k}(u))$$

or with obvious notation:

$$c_2(u,k) = \kappa_4 \cdot \sum_{j=-\infty}^{\infty} \alpha_j^2(u)\alpha_{j+k}^2(u) + 2 \cdot c^2(u,k)\,.$$

$$c_2(u,k) = \kappa_4 \cdot \sum_{j=-\infty}^{\infty} \alpha_j^2(u)\alpha_{j+k}^2(u) + 2\cdot c^2(u,k)$$

Taking the sum over all $k \in \mathbb{Z}$, using the fact that

$$\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \alpha_j^2(u)\alpha_{j+k}^2(u) = c^2(u,0)/\sigma^4\,,$$

recall that $\mathrm{Var}(\varepsilon_t) = \sigma^2$, and integrating over time, we obtain

$$\eta_4 = \Big(\int_0^1 c^2(u,0)du\Big)^{-1}\sum_{k=-\infty}^{\infty}\Big(\int_0^1 c_2(u,k)du - 2\int_0^1 c^2(u,k)du\Big)$$

$$= \frac{2\pi\int_0^1 f_{X^2}(u,0)\,du - 4\pi\int_{-\pi}^{\pi}\int_0^1 f_X^2(u,\lambda)\,du\,d\lambda}{\int_0^1\Big(\int_{-\pi}^{\pi} f_X(u,\lambda)d\lambda\Big)^2 du}\,,$$

f_{X^2} denotes the spectral density of the squared process $\{X_t^2(u)\}$.

Estimation of fourth order cumulant

The representation

$$\eta_4 = \frac{2\pi \int_0^1 f_{X^2}(u,0)\,du - 4\pi \int_{-\pi}^{\pi} \int_0^1 f_X^2(u,\lambda) du\, d\lambda}{\int_0^1 \left(\int_{-\pi}^{\pi} f_X(u,\lambda)\,d\lambda \right)^2 du},$$

now allows for construction of a consistent estimator $\widehat{\eta}_4$ of η_4.

Bootstrap Algorithm

Bootstrap algorithm to generate time domain pseudo-observations $X^*_{1,n}, X^*_{2,n}, \ldots, X^*_{n,n}$ of the locally stationary process $\mathbf{X}_n$.

I: Let $\widetilde{\eta}_4 = \widehat{\eta}_4 + 3 > 0$, where $\widehat{\eta}_4$ is a consistent estimator of η_4.

II: Generate i.i.d. pseudo innovations $\varepsilon^*_1, \ldots, \varepsilon^*_n$ according to

$$P(\varepsilon^*_t = \sqrt{\widetilde{\eta}_4}\ ) = P(\varepsilon^*_t = -\sqrt{\widetilde{\eta}_4}\ ) = \frac{1}{2\widetilde{\eta}_4}, \quad P(\varepsilon^*_t = 0) = 1 - \frac{1}{\widetilde{\eta}_4}.$$

III: Calculate the finite Fourier transform of $\varepsilon^*_1, \ldots, \varepsilon^*_n$: $J^*_{n,\varepsilon}(\lambda) = (2\pi n)^{-1/2} \sum_{t=1}^n \varepsilon^*_t e^{-i\lambda t}$

IV: Generate pseudo-observations $X^*_{1,n}, X^*_{2,n}, \ldots, X^*_{n,n}$ by

$$X^*_{t,n} = \sqrt{\frac{2\pi}{n}} \sum_{\lambda_j} \widehat{f}^{1/2}\left(\frac{t}{n}, \lambda_j\right) J^*_{n,\varepsilon}(\lambda_j) e^{it\lambda_j},$$

where $\widehat{f}(u, \lambda)$ is an estimator of $f(u, \lambda)$.

Bootstrap Algorithm

The bootstrap pseudo series $\{X^*_{t,n}, t = 1, 2, \ldots, n\}$ generated by means of the above algorithm (asymptotically) correctly mimics the first and second order local moment structure of the underlying stochastic process.

First: Conditionally on $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$, we have

$$E^*(X^*_{t,n}) = 0 \text{ for all } t\,.$$

Bootstrap Algorithm

Next: For any given $k \in \mathbb{Z}$ and n large,

$$\begin{aligned}
&\mathrm{Cov}(X^*_{t,n}, X^*_{t+k,n}) \\
&= \frac{1}{n^2} \sum_{\lambda_j, \lambda_s} \widehat{f}^{1/2}\left(\frac{t}{n}, \lambda_j\right) \widehat{f}^{1/2}\left(\frac{t+k}{n}, \lambda_s\right) e^{it\lambda_j - i(t+k)\lambda_s} \sum_{r=1}^{n} e^{ir(\lambda_j - \lambda_s)} \\
&= \frac{1}{n} \sum_{\lambda_j} \widehat{f}^{1/2}\left(\frac{t}{n}, \lambda_j\right) \widehat{f}^{1/2}\left(\frac{t+k}{n}, \lambda_j\right) e^{ik\lambda_j} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{f}\left(\frac{t}{n}, \lambda\right) e^{-ik\lambda} d\lambda + O_P(kn^{-1} + n^{-1}) \\
&= c\left(\frac{t}{n}, k\right) + o_P(1).
\end{aligned}$$

Integrated Preperiodogram

Moreover, the pseudo time series $\{X^*_{t,n}, t = 1, 2, \ldots, n\}$ correctly mimics the weak and asymptotically vanishing covariance of periodogram ordinates at different frequencies including the correct constant of the leading term, if $\widehat{\eta}_4$ is consistent for η_4.

Integrated Preperiodogram

Several statistics of interest in inferring properties of the locally stationary processes can be written as functionals of the preperiodogram.

In their general form such statistics are given by

$$\frac{1}{n}\sum_{t=1}^{n}\int_{-\pi}^{\pi}\phi\left(\frac{t}{n},\lambda\right) I_n\left(\frac{t}{n},\lambda\right)\,d\lambda\,,$$

where ϕ is some appropriately chosen function.

Integrated Preperiodogram

In the following we are interested in estimating the distribution of

$$E_n(\phi) = \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n}\int_{-\pi}^{\pi}\phi(t/n,\lambda)I_n(t/n,\lambda)\,d\lambda - \int_0^1\int_{-\pi}^{\pi}\phi(u,\lambda)f(u,\lambda)\,d\lambda\,du\right).$$

Integrated Preperiodogram

Dahlhaus and Polonik (2009) have shown under regularity conditions, that, as $n \to \infty$, $E(E_n(\phi)) \to 0$ and $\mathrm{Var}(E_n(\phi)) \to \tau^2$, where

$$\begin{aligned} \tau^2 &= 4\pi \int_0^1 \int_{-\pi}^{\pi} \phi(u,\lambda)\phi(u,-\lambda) f^2(u,\lambda)\, d\lambda\, du \\ &\quad + \eta_4 \int_0^1 \Big(\int_{-\pi}^{\pi} \phi(u,\lambda) f(u,\lambda)\, d\lambda \Big)^2 du\,. \end{aligned}$$

Furthermore, they established that

$$\mathcal{L}(E_n(\phi)) \;\Rightarrow\; N(0,\tau^2) \quad \text{as} \;\; n \to \infty\,,$$

Bootstrapping the Integrated Preperiodogram

Now, to approximate the distribution of the random sequence $(E_n(\phi), n \in \mathbb{N})$ we propose to use the bootstrap analogue

$$E_n^*(\phi) = \sqrt{n}\left(\frac{1}{n}\sum_{t=1}^{n}\int_{-\pi}^{\pi}\phi(t/n,\lambda)I_n^*(t/n,\lambda)\,d\lambda - \int_0^1\int_{-\pi}^{\pi}\phi(u,\lambda)\widehat{f}(u,\lambda)\,d\lambda\,du\right),$$

where

$$I_n^*\left(\frac{t}{n},\lambda\right) = \frac{1}{2\pi}\sum_{k:1\leq[t+1/2\pm k/2]\leq n} X^*_{[t+1/2+k/2],n}X^*_{[t+1/2-k/2],n}e^{-ik\lambda},$$

denotes the preperiodogram based on the bootstrap pseudo series $X^*_{1,n}, X^*_{2,n}, \ldots, X^*_{n,n}$.

Bootstrapping the Integrated Preperiodogram

We can prove the following result (K. and Paparoditis (2012))

Theorem

Suppose that suitable regularity conditions are fulfilled. Then, in probability, and as $n \to \infty$, we have that

(i) $E^*(E_n^*(\phi)) = o_P(1)$.

(ii) $\mathrm{Var}^*(E_n^*(\phi)) = \tau^2 + o_P(1)$, *where* τ^2 *is as above and,*

(iii) $\mathcal{L}(E_n^*(\phi)|X_{1,n}, X_{2,n}, \ldots, X_{n,n}) \;\Rightarrow\; N(0, \tau^2)$.

Related work: Kirch and Politis (2011).

Thank you very much for listening!